Continuum Mechanics

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Abstract

Full Text

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CONSTRUCTION OF NONHOLONOMIC MODELS OF CONTINUOUS MEDIA WITH ALLOWANCE FOR FINITE DEFORMATIONS AND CERTAIN PHYSICOCHEMICAL EFFECTS

In many modern practical problems, the description of the mechanical behavior of materials is inseparable from thermal and other physicochemical properties. Such phenomena as the release or absorption of heat by a body during its deformations and the influence of temperature on the mechanical characteristics of bodies, the appearance of residual deformations upon removal of internal stresses, the phenomena of creep and stress relaxation, the phenomenon of high elasticity, etc., play an important role in the solution of many technical problems. These and many other effects manifest themselves especially strongly in polymeric materials.

Below we set forth new theoretical approaches that make it possible to construct models of continuous media for theoretical studies and for the formulation and processing of experiments. We shall carry out the construction of such models within the framework of a geometrically and dynamically nonlinear theory of finite deformations, as applied to the phenomena listed above.

Let ξ^1, ξ^2, ξ^3 be some system of Lagrangian coordinates embedded in the medium. For the distance element ds_0 between points of the medium with coordinates ξ^i and $\xi^i + d\xi^i$ in the initial state, we have:

$$ds_0^2 = {}^0g_{\alpha\beta} d\xi^\alpha d\xi^\beta, \tag{1}$$

where ${}^0g_{\alpha\beta}$ are the components of the fundamental tensor. In the deformed state the element ds_0 passes into ds, for which one may write:

$$ds^2 = \hat{g}_{\alpha\beta} d\xi^\alpha d\xi^\beta. \tag{2}$$

The covariant components of the finite-deformation tensor are defined by the formulas:

$$\varepsilon_{ij} = \frac{1}{2}\left(\hat{g}_{ij} - {}^0g_{ij}\right). \tag{3}$$

Let $\rho(\xi^1, \xi^2, \xi^3, t)$ denote the density of a material particle. The continuity equation gives:

$$\rho\sqrt{g} = \rho_0\sqrt{{}^0g}, \tag{4}$$

where ρ_0 is the density in the initial state, and g denotes the determinant $|g_{ik}|$ and, respectively, ${}^0g = |{}^0g_{ik}|$.

Let p^{ij} be the contravariant components of the stress tensor in the corresponding coordinate system. Denote by $\mathbf{P}^i$ the total internal force acting on the corresponding movable plane element of area $\omega(t)$, defined by the differentials $d\xi^k$ and $d\xi^j$ $(k, j \neq i)$. It is easy to verify the formulas:

$$\mathbf{P}^i = P^{i\alpha}\hat{\mathbf{e}}_\alpha = \rho_0\sqrt{{}^0g}\,\omega^0\sigma^{i\alpha}\hat{\mathbf{e}}_\alpha, \qquad \text{where } \sigma^{ij} = \frac{p^{ij}}{\rho}; \tag{5}$$

here $\overset{0}{\omega} = \omega(t_0)$ and $\hat{\mathfrak{e}}_\alpha$ are the basis vectors for the accompanying coordinate system ($\hat{\mathfrak{e}}_\alpha = \partial\mathbf{r}/\partial\xi^\alpha$, $\mathbf{r}(\xi^i, t)$ is the radius vector of points of the medium).

For the elementary work of internal surface forces, referred to unit mass, the formula is valid:

$$dA^{(i)} = -\sigma^{\alpha\beta}d\varepsilon_{\alpha\beta} \quad (\sigma^{\alpha\beta} = p^{\alpha\beta}/\rho). \tag{6}$$

As is known, in the classical theory of elasticity, for isothermal processes the quantities ε_{ij} completely determine the quantities p^{ij} and σ^{ij}; however, in processes of creep, relaxation, and in many other cases, for one and the same deformation*, determined by ε_{ij}, there may be different stresses p^{ij} or σ^{ij}.

Bearing such phenomena in mind, in what follows we shall assume that the quantities ε_{ij} and σ^{ij} are, in the general case, not connected by any finite relations independent of the values of other variable quantities determining the state of a small particle. As other quantities determining the state of a particle, we shall take the temperature** $T(\xi^i, t)$ and a certain parameter $\chi(\xi^i, t)$.

The explicit introduction of the variable parameter χ, connected with a quantitative description of internal physico-chemical processes essential for mechanics, is the principal difference of the macroscopic theory being developed. The introduction of several such parameters is already a technique introduced into theory in gas dynamics when reversible or irreversible chemical reactions are taken into account, when ionization is taken into account, in the theory of lagging heat capacities, etc. Analogous parameters are introduced in the mechanics of solids when magnetic and electrical properties, phase transitions, recrystallization, etc. are taken into account. Observations and descriptions of creep phenomena and many features of the properties of polymers urgently require the explicit introduction of such parameters.

To simplify the theory, and also in view of the great possibilities of describing a variety of different effects with a single parameter χ, in what follows we shall introduce only one parameter χ. The explicit use of such a parameter in the macroscopic equations will make it possible to explain experimental data and to move toward solving the most important problem: identifying the mechanism of the corresponding microscopic molecular processes and their quantitative description.

Thus, let us consider models of bodies for which the state of small elements and the corresponding thermodynamic functions are determined by the system of parameters:

$$\varepsilon_{ij}, \quad T, \quad \chi, \quad \sigma^{ij} = p^{ij}/\rho. \tag{7}$$

In addition to the system of parameters (7), generally variable, the physical properties of the medium may also be determined by a system of physical constants—scalars or tensors.

Let S and F denote the entropy and free energy of a small particle, referred to unit mass.

The first and second laws of thermodynamics may be written in the form:

$$dF - \sigma^{\alpha\beta} d\varepsilon_{\alpha\beta} + S\,dT = -dQ', \qquad T\,dS = dQ + dQ', \tag{I}$$

where dQ is the external influx of heat to the particle, referred to unit mass, and dQ' is uncompensated heat. Here we assume that the external influx of energy reduces to the elementary work of external forces (mass and surface), which appear in the macroscopic dynamical equations of a continuous medium, and to the influx of thermal energy dQ.

For reversible processes $dQ' = 0$, and for irreversible ones $dQ' \geq 0$. Further, as the simplest hypothesis, let us put

$$dQ' = \varkappa \frac{d\chi}{dt}\, d\chi, \qquad \varkappa > 0, \tag{8}$$

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[*] In particular, for $p^{ij} = 0$ different residual deformations are possible. Their appearance is not always connected with irreversible effects. For example, in an ideal or viscous liquid the appearance of residual deformations may arise reversibly.

[**] In what follows only such nonequilibrium states are considered for which the temperature can be defined.

where $\varkappa$ is a scalar function of the parameters (7). Hypothesis (8) can be modified on the basis of a more detailed study of the properties of particular bodies.

In forming the increment dF in equation (1), in accordance with (7), we shall assume that

$$F = F\left(\overset{0}{g}_{ij},\ \varepsilon_{ij},\ \sigma^{ij},\ T,\ \chi,\ \text{physical constants}\right). \tag{9}$$

The explicit introduction of the dependence of the free energy F on the values σ^{ij} is associated with the assumption that these quantities may be independent of the values ε_{ij}, T, and χ. This circumstance introduces an essential distinction between the models considered below and the models of elastic bodies in the usual sense.

Equation (1), taking (8) and (9) into account, can be written in the form

$$\frac{\partial F}{\partial \chi}\,d\chi + \frac{\partial F}{\partial \sigma^{ij}}\,d\sigma^{ij} + \left(\frac{\partial F}{\partial \varepsilon_{\alpha\beta}} - \sigma^{\alpha\beta}\right) d\varepsilon_{\alpha\beta} + \left(\frac{\partial F}{\partial T} + S\right) dT = -\varkappa \frac{d\chi}{dt}\,d\chi. \tag{10}$$

In equality (10), the differentials $d\sigma^{ij}$ are taken for a given particle ($\xi^i = \text{const}$) in the accompanying Lagrangian coordinate system. The relation between the various time derivatives of tensors is revealed in paper (1). From the proposed arguments and certain other considerations it follows that the use of the derivatives $d\sigma^{ij}/dt$ is natural and convenient. In paper (1) an expression is given for $d\sigma^{ij}/dt$ through the derivatives of the stress tensor in a fixed reference frame and through the elements of the motion of the medium.

If, along with the independence of the parameters (7), it is assumed that the quantities $d\chi$, $d\varepsilon_{\alpha\beta}$, dT, and $d\sigma^{ij}$ are also independent, then we obtain a system of finite equations $\partial F/\partial \sigma^{ij} = 0$, which makes it possible to reduce the number of independent defining parameters in (7); therefore, the preservation of assumption (7) can be associated with the presence of nonholonomic relations:

$$d\sigma^{ij} = A^{ij}\,d\chi + B^{ij}\,dT + A^{ij\alpha\beta}\,d\varepsilon_{\alpha\beta}, \tag{11}$$

where the increments $d\chi$, dT, $d\varepsilon_{\alpha\beta}$ are regarded as linearly independent, and the components of the tensors B^{ij} and $A^{ij\alpha\beta}$ as functions of the system of parameters (7). In this case the components of the tensors A^{ij} may be regarded not only as functions of the system of parameters (7), but also as functions of their derivatives with respect to time and to the coordinates, while A^{ij} have finite values when $d\chi = 0$.

Relations (10) and (11) lead to the following equalities, which may be regarded as a generalization of the equations of state of the usual theory of elasticity:

$$\frac{\partial F}{\partial \sigma^{ij}} A^{ij\alpha\beta} + \frac{\partial F}{\partial \varepsilon_{\alpha\beta}} - \sigma^{\alpha\beta} = 0 \qquad (\alpha, \beta = 1, 2, 3); \tag{12}$$

$$\frac{\partial F}{\partial \sigma^{ij}} B^{ij} + \frac{\partial F}{\partial T} + S = 0 \tag{13}$$

and, as a kinetic equation for determining the parameter χ:

$$\frac{\partial F}{\partial \chi} + \frac{\partial F}{\partial \sigma^{ij}} A^{ij} = -\varkappa \frac{d\chi}{dt}. \tag{14}$$

For isothermal processes, when $dT = 0$, and when $d\chi = 0$, the nonholonomic relations (11) have a reversible character. Relations of this kind and the corresponding models of generalized nonholonomic elastic

bodies (hypoelastic bodies) were considered by Jaumann (^{2,3}), Truesdell (^{4}), Noll (^{5}), Prager (^{6}), Bernstein (^{7}), and others. However, these authors did not rely on the laws of thermodynamics and did not consider the free energy; therefore, in their works the six relations (12), imposed on the components $A^{ij\alpha\beta}$ as functions of ε_{ij} and σ^{ij}, are absent.

The equation of the second law $T\,dS = dQ + dQ'$, relations (11) and (14), together with the dynamic equations and equations (3) and (4), can form a closed system of equations determining a continuum model for a broad class of mechanical, thermal, and physical processes, if the quantities A^{ij}, B^{ij}, $A^{ij\alpha\beta}$, and F are fixed, with relations (12) being satisfied identically, while relation (13) can serve for calculating the entropy.

Let us now dwell briefly on one simplest variant of an isotropic body. If, just as in deriving Hooke’ s law, it is assumed that the tensor $A^{ij\alpha\beta}$ does not depend on ε_{ij} and σ^{ij}, then we obtain:

$$A^{ij\alpha\beta} = \lambda \overset{0}{g}{}^{ij} \overset{0}{g}{}^{\alpha\beta} + \mu \left(\overset{0}{g}{}^{i\alpha} \overset{0}{g}{}^{j\beta} + \overset{0}{g}{}^{j\alpha} \overset{0}{g}{}^{i\beta} \right), \tag{15}$$

where λ and μ are scalars that may be regarded as functions of T and χ.

If, in addition, it is assumed that for $\chi = \text{const}$ the thermal effects appear in the same way as in ordinary thermoelasticity, then we obtain:

$$B^{ij} = \left[\frac{\partial \lambda}{\partial T} I_1(\varepsilon) - \alpha \right] \overset{0}{g}{}^{ij} + 2 \frac{\partial \mu}{\partial T} \varepsilon_{\alpha\beta} \overset{0}{g}{}^{\alpha i} \overset{0}{g}{}^{\beta j}, \tag{16}$$

where $\alpha(T, \chi)$ determines the coefficient of thermal expansion, $I_1(\varepsilon)$ is the first invariant of the strain tensor ($I_1 = \overset{0}{g}{}^{\alpha\beta} \varepsilon_{\alpha\beta}$).

On the basis of (15) and (16), relations (11) can be written in the form:

$$d\sigma^{ij} = A^{ij}\,d\chi + d_{\chi=\text{const}}\overset{*}{\sigma}^{ij}, \tag{17}$$

where $\overset{*}{\sigma}^{ij}$ are the components of a tensor determined as a function of ε_{ij} and T according to the laws of linear thermoelasticity for σ^{ij}. In this case, relations (12) and (13) are satisfied identically if one sets

$$F = \sigma^{\alpha\beta}\varepsilon_{\alpha\beta} - \mu I_2(\varepsilon) - \frac{\lambda}{2}I_1^2(\varepsilon) + f(T,\chi);$$

$$S = -\frac{1}{2}\left[\frac{\partial\lambda}{\partial T}I_1^2 + 2\frac{\partial\mu}{\partial T}I_2\right] + \alpha I_1(\varepsilon) - \frac{\partial f}{\partial T},$$

where $I_2(\varepsilon)$ is the second invariant:

$$I_2 = \overset{0}{g}{}^{\alpha\beta}\overset{0}{g}{}^{\gamma\delta}\varepsilon_{\alpha\gamma}\varepsilon_{\beta\delta}.$$

Equations (17) and (14), for infinitesimal deformations, pass into equations of the type of the flow theory in the theory of creep (^{8}), if one sets, for example, $A^{ij} = \xi I_1(\sigma)\overset{0}{g}{}^{ij} + 2\eta\sigma^{ij}$, $-\varkappa = [\partial F/\partial\chi + A^{ij}\varepsilon_{ij}]/\varkappa_0$, where ξ, η, and $\varkappa_0$ are suitably chosen scalar functions of T, χ, σ^{ij}, and ε_{ij}.

The arguments presented above can be applied, in particular, for the thermodynamic substantiation of existing theories of creep that operate with nonholonomic models of continua.

Further study of various models and investigation of the possibility of describing specific materials within the framework of this theory will be given in subsequent works.

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Note: Figure translations are in progress. See original paper for figures.