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Abstract

Full Text

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VARIABILITY OF SIMPLE DOUBLE POINTS OF REAL PLANE ALGEBRAIC CURVES*

(Presented by Academician I. G. Petrovskii, 20 X 1961)

1. We shall adopt the following notation and terminology: $(x : y : z)$ are the coordinates of a point in the complex projective plane $*R^2$; R^2 is the real part of $*R^2$; E^2 is the complex affine plane obtained from $*R^2$ by putting $z = 1$; E^2 is the real part of $*E^2$. Suppose that a unitary metric is fixed in $*E^2$, and consequently a Euclidean metric in E^2 . $\rho^*(a^0)$ denotes the ρ -neighborhood of the point $a^0 \in *E^2$ in $*E^2$; $\rho(a^0)$, the ρ -neighborhood of the point $a^0 \in E^2$ in E^2 .

Let, by the equation

$$H(x, y, z) \equiv \sum_{\alpha+\beta+\gamma=m} A_{\alpha\beta\gamma} x^\alpha y^\beta z^\gamma = 0 \quad (1)$$

there be given a real algebraic curve of order m in the plane $*R^2$. Such a curve corresponds one-to-one to a "point" of the real projective space R^N ($A_{00} : A_{10} : A_{01} : \dots : A_{0m}$) of dimension

$$N = \frac{m(m+3)}{2}$$

(({}^{\sim}\{2\}), p. 75).

The terms "point" and "curve" in R^N will be put in quotation marks, in distinction from the same terms in $*R^2$. The curve H will also be called the "point" H in R^N . Let $E_{\alpha_0\beta_0}^N$ denote the affine space obtained from R^N by setting the coefficient $A_{\alpha_0\beta_0}$ equal to one. We regard $E_{\alpha_0\beta_0}^N$ as a Euclidean space. $S(F, \varepsilon)$ denotes the ε -neighborhood of the "point" $F \in E_{\alpha_0\beta_0}^N$ in $E_{\alpha_0\beta_0}^N$. A curve F having δ simple double points and having no other singular points in $*R^2$ will be called a δ -simple curve (or, more briefly, a simple curve).

2. **Definition of the variability of simple double points.** Let the curve

$$F \equiv \sum_{0 \leq \alpha + \beta \leq m} A_{\alpha\beta}^0 x^\alpha y^\beta = 0$$

be a δ -simple curve, all the singular points of which belong to ${}^*E^2$. Let $b_j^0(x_j^0, y_j^0)$ ($j = 1, 2, \dots, (k + q)$) be real simple double points, and let $a_\nu^0(x_\nu^0, y_\nu^0)$ and \bar{a}_ν^0 ($\nu = 1, 2, \dots, (l + s)$) be pairs of imaginary conjugate simple double points of the curve F ($k + q + 2(l + 1) = \delta$). We shall call k real points b_j^0 ($j = 1, 2, \dots, k$) and l pairs of imaginary conjugate points a_ν^0 and \bar{a}_ν^0 ($\nu = 1, 2, \dots, l$) $v(\varepsilon, \rho)$ -variable (or, more briefly, variable), if, for some coefficient $A_{\alpha_0\beta_0}$ ($A_{\alpha_0\beta_0}^0 \neq 0$), for any sufficiently small $\varepsilon > 0$ and $\rho > 0$ one can find such a $v(\varepsilon, \rho) > 0$ that, choosing arbitrarily points $b_j \in v(b_j^0)$ ($j = 1, 2, \dots, k$), $a_\nu \in v^*(a_\nu^0)$ and $\bar{a}_\nu \in v^*(\bar{a}_\nu^0)$ ($\nu = 1, 2, \dots, l$), one can find a curve

$$G = \sum_{0 \leq \alpha + \beta \leq m} \tilde{A}_{\alpha\beta} x^\alpha y^\beta = 0$$

* We have already considered this question in (3).

and such a continuous "curve" $FG \subset S(F, \varepsilon) \subset E_{\delta_0\beta_0}^\gamma$, that:

- 1) every curve $H \in FG$ is a δ -simple curve;
- 2) the ordinary double points of the curve H are located one in each of the neighborhoods $v(b_j^0)$ ($j = 1, 2, \dots, k$), $v^*(a_\nu^0)$ and $v^*(\bar{a}_\nu^0)$ ($\nu = 1, 2, \dots, l$), $\rho(b_j^0)$ ($j = (k + 1), \dots, (k + q)$), $\rho^*(a_\nu^0)$ and $\rho^*(\bar{a}_\nu^0)$ ($\nu = (l + 1), \dots, (l + s)$) (pairwise nonintersecting);
- 3) the coordinates of the singular points of the curve H are continuous functions of the coefficients of the curve H , if $H \in FG$;
- 4) the curve G has singular points at the chosen points b_j ($j = 1, 2, \dots, k$), a_ν and \bar{a}_ν ($\nu = 1, 2, \dots, l$), and one singular point in the ρ^* -neighborhood of each of the remaining singular points of the curve F^* .

Lemma 1 . *If the system of $(3k + 6l + q + 2s)$ linear equations with respect to all coefficients $A_{\alpha\beta}$ of the curve H*

$$H(b_j^0) = 0, \quad H_x(b_j^0) = 0, \quad H_y(b_j^0) = 0 \quad (j = 1, 2, \dots, k); \quad (2^1)$$

$$H(a_\nu^0) = 0, \quad H_x(a_\nu^0) = 0, \quad H_y(a_\nu^0) = 0,$$

$$H(\bar{a}_\nu^0) = 0, \quad H_x(\bar{a}_\nu^0) = 0, \quad H_y(\bar{a}_\nu^0) = 0 \quad (\nu = 1, 2, \dots, l); \quad (2^2)$$

$$H(b_j^0) = 0 \quad (j = (k + 1), \dots, (k + q)); \quad (2^3)$$

$$H(a_\nu^0) = 0, \quad H(\bar{a}_\nu^0) = 0 \quad (\nu = (l + 1), \dots, (l + s)) \quad (2^4)$$

is linearly independent, then the k points $b_j^{(0)}$ ($j = 1, 2, \dots, k$) and the $2l$ points a_j^0 and \bar{a}_ν^0 ($\nu = 1, 2, \dots, l$) of the curve F are variable.

Theorem 1. For the variability of k real and l pairs of imaginary conjugate ordinary double points of the curve F , it is necessary that the inequality

$$3(k + 2l) + (q + 2s) \leq N. \quad (3)$$

Lemma 2. Let the curve F be irreducible, $l = 0$, and let the k ordinary double points b_j^0 ($j = 1, 2, \dots, k$) of the curve F be variable. Then the indicated k ordinary real double points of the curve F can be displaced (arbitrarily little) to such points b_j ($j = 1, 2, \dots, k$) that the system of equations

$$H(b_j) = 0, \quad H_x(b_j) = 0, \quad H_y(b_j) = 0 \quad (j = 1, 2, \dots, k) \quad (4)$$

(with respect to the coefficients $A_{\alpha\beta}$ of the curve H) will be linearly independent.

3. Let the curve F be irreducible. Consider the linear series g_n^r , cut out on F by the linear system of curves of order m determined by the system of equations (2).

Denote $k + 2l = h$, $q + 2s = g$. We introduce a single notation for all singular points of the curve F . Let c_j^0 ($j = 1, 2, \dots, h$) be the ordinary double points whose variability is under study, and let c_j^0 ($j = (h + 1), \dots, (h + g)$) be the remaining singular points of the curve F . Let P_j and Q_j ($j = 1, 2, \dots, (h + g)$) be the branches of the curve F with center at the point c_j^0 . Consider the cycles

$$D = \sum_{j=1}^{h+g} (P_j + Q_j), \quad B = \sum_{j=1}^h (P_j + Q_j).$$

* In (3) we gave the definition of variability for the curve F having only real ordinary double points.

** Lemma 1 is a generalization of Theorem 7 (3).

It is known that the series g_n^r is the residual of the series cut out on the curve F by all curves of order m , with respect to the cycle $D + B$, and g_n^r is a complete series ((²), p. 211, vol. 6.3).

In what follows we shall assume that condition (3) is satisfied (otherwise the question is settled by Theorem 1). Since F is irreducible, we have

$$h + g \leq \frac{m(m-3)}{2} + 1. \quad (5)$$

Adding (3) and (5), we obtain

$$4h + 2g \leq m^2 + 1. \quad (6)$$

But the series g_n^r has order $n = m^2 - (4h + 2g)$; therefore n can be less than zero only for a curve of genus $p = 0$.

Theorem 2*. Let the curve F have genus $p = 0$. Then, for the variability of any k real and l pairs of imaginary conjugate ordinary double points of the curve F , it is necessary and sufficient that the inequality

$$2k + 4l \leq 3m - 1. \quad (7)$$

hold.

Lemma 3. Let the curve F have genus $p \geq 1$. Then, for the linear independence of the system (2), it is necessary and sufficient that the series g_n^r be nonspecial.

Theorem 3. Let the curve F have genus $p \geq 1$. Then, for the variability of any of its k real and l pairs of imaginary conjugate ordinary double points, it is sufficient that inequality (7) hold.

Lemma 4. Let the curve F have genus $p \geq 1$ and suppose that the condition

$$3(k + 2l) + (q + 2s) = N \quad (8)$$

is satisfied.

Then the system (2) is linearly dependent for any choice of k real and l pairs of imaginary conjugate ordinary double points of the curve F .

Theorem 4. Let the curve F have genus $p \geq 1$. Then, if the conditions

$$l = 0, \quad 3k = N, \quad (9)$$

are satisfied, the k real double points of the curve F are not variable**.

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* Theorem 2 is a strengthening and generalization of Theorem 8 from ⁽³⁾.

** The well-known special case is $m = 6$, $k = 9$, $p = 1$ ⁽¹⁾.

Note: Figure translations are in progress. See original paper for figures.

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