MATHEMATICAL PHYSICS

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Abstract

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MODELS WITH A FINITE NUMBER OF DEGREES OF FREEDOM FOR A CERTAIN CLASS OF PROBLEMS OF MATHEMATICAL PHYSICS

(DIFFERENCE SCHEMES WITH A CONSERVATION LAW)

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At the seminar on questions of turbulence in 1958–1959, A. N. Kolmogorov posed the following problem: to devise a finite-dimensional model of the Navier–Stokes equations for the motion of a viscous incompressible fluid. The model must, when the number of degrees of freedom tends to infinity, pass over into the viscous Navier–Stokes fluid and possess a law of conservation (dissipation) of energy.

A solution of this problem is given by the author in terms of difference schemes. For simplicity we consider the case of plane motion. Then the motion of a viscous fluid is described by means of the stream function $\psi(x, y; t)$, satisfying the vorticity equation

$$\frac{\partial}{\partial t}\Delta\psi + \frac{\partial(\psi, \Delta\psi)}{\partial(x, y)} = \nu\Delta^2\psi, \qquad x, y \in \Omega, \tag{1}$$

with the no-slip boundary condition

$$\psi\big|_\Gamma = \frac{\partial\psi}{\partial n}\bigg|_\Gamma = 0 \tag{2}$$

on the fixed boundary Γ of the domain Ω. Sometimes one has to consider a solution on the whole x, y-plane which is periodic in both variables; then (2) must be replaced by

$$\begin{gathered}
\psi\Big|_{x=-a} = \psi\Big|_{x=a}, \qquad \frac{\partial\psi}{\partial y}\Big|_{x=-a} = \frac{\partial\psi}{\partial y}\Big|_{x=+a}, \\
\frac{\partial^2\psi}{\partial y^2}\Big|_{x=-a} = \frac{\partial^2\psi}{\partial y^2}\Big|_{x=a}, \qquad \frac{\partial^3\psi}{\partial y^3}\Big|_{x=-a} = \frac{\partial^3\psi}{\partial y^3}\Big|_{x=a}, \\
\psi,\ \frac{\partial\psi}{\partial x},\ \frac{\partial^2\psi}{\partial x^2},\ \frac{\partial^3\psi}{\partial x^3}\Big|_{y=-b} = \psi,\ \frac{\partial\psi}{\partial x},\ \frac{\partial^2\psi}{\partial x^2},\ \frac{\partial^3\psi}{\partial x^3}\Big|_{y=b}.
\end{gathered} \tag{2'}$$

One may also consider the more general case of a moving boundary, or one having a period in x and being rectilinear in y, etc. We shall set forth a number of general constructions that led us to the solution of Kolmogorov’ s problem; namely, we shall construct difference operators of mathematical physics possessing the properties of differential operators with respect to integral properties. In this sense the present work is closely connected with the works of V. I. Lebedev (^{1}), who investigated from this point of view the difference approximation of Sobolev’ s equation in Cartesian coordinates. For simplicity of exposition we restrict ourselves to the plane case and to an orthogonal curvilinear coordinate system in which the line element is expressed by the formula $ds^2 = (g^1)^2\,dp_1^2 + (g^2)^2\,dp_2^2$, and the area element by $d\sigma = g^1 g^2\,dp_1 dp_2$.

We first consider the case of an unbounded domain (the whole (p_1, p_2)-plane). Take a square mesh (if the nature of the problem requires a nonuniform mesh, one may introduce new coordinates p_1', p_2', in which the mesh can be taken uniform); therefore the step will everywhere be···

to be regarded as unitary. The first-order differential operators of mathematical physics in an orthogonal coordinate system have the form

$$\operatorname{grad}\psi = \left\{\frac{1}{g^1}\frac{\partial\psi}{\partial p_1},\ \frac{1}{g^2}\frac{\partial\psi}{\partial p_2}\right\},$$

$$\begin{aligned}
\operatorname{div}\mathbf{A} &= \frac{1}{g^1 g^2}\left[\frac{\partial}{\partial p_1}(g^2 A^1) + \frac{\partial}{\partial p_2}(g^1 A^2)\right], \\
\operatorname{rot}^3\mathbf{A} &= \frac{1}{g^1 g^2}\left[\frac{\partial}{\partial p_1}(g^2 A^2) - \frac{\partial}{\partial p_2}(g^1 A^1)\right], \\
\operatorname{rot}\psi &= \left\{\frac{1}{g^2}\frac{\partial}{\partial p_2}\psi,\ -\frac{1}{g^1}\frac{\partial}{\partial p_1}\psi\right\},
\end{aligned} \tag{3}$$

the operation $\operatorname{rot}^3\mathbf{A}$ is applied to vectors $\mathbf{A}(p_1, p_2)$ and transforms them into scalar functions (into vectors perpendicular to the plane (p_1, p_2)); the operation $\operatorname{rot}\psi$ is applied to scalars $\psi(p_1, p_2)$ (to vectors perpendicular to the plane (p_1, p_2)) and transforms them into vectors lying in the plane (p_1, p_2). The most important second-order operator is the Laplacian Δ, applied to scalars,

$$\Delta\psi = \operatorname{div}\operatorname{grad}\psi = -\operatorname{rot}^3\operatorname{rot}\psi = \frac{1}{g^1 g^2}\left[\frac{\partial}{\partial p_1}\left(\frac{g^2}{g^1}\frac{\partial\psi}{\partial p_1}\right) + \frac{\partial}{\partial p_2}\left(\frac{g^1}{g^2}\frac{\partial\psi}{\partial p_2}\right)\right]$$

(see, for example, (2)). Starting from formula (3), we shall write the difference operators of mathematical physics. Number the points of the star as shown in Fig. 1; functions will be assumed specified at points for which the sum of the indices is an integer (in the figure they are shown by circles), while the values of first-order differential operators are at points with a half-integer sum of indices (shown by crosses). Second-order operations, as well as the functions themselves, are defined at integer points, etc. Formulas analogous to formulas (3), naturally, have the form:

$$\operatorname{grad}_h\psi_{00} = \left\{\frac{1}{g^1_{00}}\left(\psi_{1/2,0} - \psi_{-1/2,0}\right),\ \frac{1}{g^2_{00}}\left(\psi_{0,1/2} - \psi_{0,-1/2}\right)\right\},$$

$$\operatorname{div}\mathbf{A}_{00} = \frac{1}{g^1_{00}g^2_{00}}\left[\left(g^2_{1/2,0}A^1_{1/2,0} - g^2_{-1/2,0}A^1_{-1/2,0}\right) + \left(g^1_{0,1/2}A^2_{0,1/2} - g^1_{0,-1/2}A^2_{0,-1/2}\right)\right],$$

$$\operatorname{rot}^3\mathbf{A}_{00} = \frac{1}{g^1_{00}g^2_{00}}\left[\left(g^2_{1/2,0}A^2_{1/2,0} - g^2_{-1/2,0}A^2_{-1/2,0}\right) - \left(g^1_{0,1/2}A^1_{0,1/2} - g^1_{0,-1/2}A^1_{0,-1/2}\right)\right], \tag{4}$$

$$\operatorname{rot}\psi_{00} = \left\{\frac{1}{g^2_{00}}\left(\psi_{0,1/2} - \psi_{0,-1/2}\right),\ -\frac{1}{g^1_{00}}\left(\psi_{1/2,0} - \psi_{-1/2,0}\right)\right\}.$$

Fig. 1

It is easy to see that the difference Laplace operator, obtained as the product of the difference operators div grad or $-\operatorname{rot}^3\operatorname{rot}$, has the form

$$\Delta_h\psi_{00} = \frac{1}{g^1_{00}g^2_{00}}\left[\frac{g^2_{1/2\,0}}{g^1_{1/2\,0}}(\psi_{10} - \psi_{00}) - \frac{g^2_{-1/2\,0}}{g^1_{-1/2\,0}}(\psi_{00} - \psi_{-10}) + \right.$$
$$\left. + \frac{g^1_{0\,1/2}}{g^2_{0\,1/2}}(\psi_{01} - \psi_{00}) - \frac{g^1_{0\,-1/2}}{g^2_{0\,-1/2}}(\psi_{00} - \psi_{0-1})\right]. \tag{5}$$

Taking into account that the area element has the form $d\sigma_{00} = g^1_{00}g^2_{00}$, it is easy to verify that (under the assumption of sufficiently strong decrease of the functions at infinity) difference analogues of various variants of Green' s formula hold:

$$\sum_{\beta}\sum_{\alpha}(\Delta_h\varphi)_{\alpha\beta}\psi_{\alpha\beta}g^1_{\alpha\beta}g^2_{\alpha\beta} = -\sum_{\beta}\sum_{\alpha}(\operatorname{grad}_h\varphi)_{\alpha\beta}(\operatorname{grad}_h\psi)_{\alpha\beta}g^1_{\alpha\beta}g^2_{\alpha\beta}; \tag{6}$$

$$\sum_{\beta}\sum_{\alpha}((\operatorname{rot}_h\psi)_{\alpha\beta},\, \mathbf{A}_{\alpha\beta})g^1_{\alpha\beta}g^2_{\alpha\beta} = -\sum_{\beta}\sum_{\alpha}\psi_{\alpha\beta}\operatorname{rot}^3\mathbf{A}_{\alpha\beta}g^1_{\alpha\beta}g^2_{\alpha\beta} \tag{7}$$

and so on. (The summation in formulas (6)-(7) is taken over all points, both integral and half-integral.) The basic equality used in proving (6), (7), and so on, has the form

$$\sum_{-\infty}^{\infty}(a_{m+1/2}-a_{m-1/2})b_m = -\sum_{-\infty}^{\infty}(b_{m+1}-b_m)a_{m+1/2}; \tag{8}$$

this equality holds for sufficiently rapidly decreasing a_m, b_m and for finite sums, if two "boundary" terms vanish. The last remark makes it possible to consider Green' s formulas (6)-(7) valid also in a finite domain, if the functions vanish in a strip of "width" equal to a pair of integral and a pair of half-integral points.

We now give the solution of A. N. Kolmogorov' s problem. To this end we shall assume that the fluid, defined by equations (1), fills a domain Ω. Cover Ω with a grid of the type indicated above and single out the set of interior grid points Ω_{2h}, having the property that the grid inside Ω contains a 13-point "star" with center at the given point (i.e., the points with coordinates $(0,0)$; $(\pm h, 0)$; $(\pm 2h, 0)$; $(0, \pm h)$; $(0, \pm 2h)$; $(\pm h, \pm h)$; $(\pm h, \mp h)$). At these points we shall write the difference approximation (1). Points not belonging to Ω_{2h} but belonging to the "stars" with centers in Ω_{2h} form a strip of boundary points; at them we impose the no-slip condition: $\psi_{mn} = 0$. At the points of Ω_{2h} the equations have the form

$$\frac{d}{dt}\Delta_h\psi_{mn} = \operatorname{rot}_h[\operatorname{rot}_h\psi_{mn}\times\widetilde{\Delta_h\psi_{mn}}] + \nu\Delta_h^2\psi_{mn}, \tag{9}$$

where $\widetilde{\Delta_h\psi_{mn}}$ is the value of $\Delta_h\psi_{mn}$, shifted in some way from the nearest integral points to the corresponding half-integral point. Equation (9), like (1), possesses the law of energy dissipation

$$\frac{1}{2}\frac{d}{dt}\sum\sum|\operatorname{rot}\psi_{mn}|^2 = -\nu\sum\sum|\Delta_h\psi_{mn}|^2,$$

whose validity follows from formulas (6)-(7).

Our method is significant not only as leading to more "physical," always stable difference equations. It makes it possible to transfer to the difference equations of mathematical physics certain methods of the theory of partial differential equations, based on obtaining a priori estimates, developed in the works of Sobolev, Friedrichs, Vishik, and others; in particular, in terms of finite dif-

the results of O. A. Ladyzhenskaya (3) can be repeated, and theorems on the existence and uniqueness of the solution in the large for equation (9) can be obtained, with estimates that make it possible to pass to the limit to equation (1).

For some problems the grid splits into "subgrids" of points of the form (m, n) and $(m+\frac{1}{2}, n+\frac{1}{2})$, i.e., into ordinary square grids with a number of points twice smaller than the original one. This applies, for example, to the Laplace equation in an arbitrary orthogonal system.

In addition to equations obtained from elementary differential operators, let us consider one very broad class of quasilinear equations introduced by S. K. Godunov (4):

$$\frac{\partial}{\partial t} L^0_{q^i} + \sum_{j=1}^{M} \frac{\partial}{\partial x_j} L^j_{q^i} = \varepsilon \Delta q^i, \qquad i = 1, 2, \dots, Q$$

(where $L^0(q^1, \dots, q^Q)$ is such that the form $(\sum_{i=1}^{Q} L^0_{q^i} \cdot q^i - L)$ is positive definite), with boundary condition $q^i = 0$ on Γ. These equations have an "entropy integral"

$$\int \left(\sum_{i=1}^{Q} L^0_{q^i} q^i - L^0 \right) dx \Big|_0^t = -\varepsilon \iint \sum_{i=1}^{Q} |\operatorname{grad} q^i|^2 \, dx \, dt.$$

Among Godunov's equations (for $\varepsilon = 0$) are variational equations, the equations of gas dynamics (including relativistic gas dynamics), Maxwell' s equations and equations of crystal optics, and many others.

Without loss of generality, consider the case of one equation with two independent variables

$$\frac{\partial A_u}{\partial t} + \frac{\partial B_u}{\partial x} = \varepsilon \frac{\partial^2 u}{\partial x^2}, \qquad u(x,t)\big|_{x=0,\,1} = 0; \qquad u(x,0) = \varphi(x).$$

We approximate the function $u(x,t)$ by its values at integer points $u_n(t)$; the function $B_u(u)$ at half-integer points by the expression

$$B_u(u)\big|_{n+1/2} = \frac{B(u_{n+1}) - B(u_n)}{u_{n+1} - u_n};$$

and the derivative $\frac{\partial B_u}{\partial x}$ at integer points by the formula

$$\frac{\partial B_u}{\partial x}\Big|_n = \frac{B_u\big|_{n+1/2} - B_u\big|_{n-1/2}}{h}.$$

Finally we shall have

$$\frac{dA_u(u_n)}{dt} + \frac{B_u\big|_{n+1/2} - B_u\big|_{n-1/2}}{h} = \varepsilon \frac{u_{n+1} - 2u_n - u_{n-1}}{h^2},$$

$$n = 1, 2, \ldots, N-1, \qquad u_0 = u_N = 0.$$

Multiplying this equation by $u_n(t)$, summing over n, and integrating with respect to t, we obtain the "conservation law"

$$\sum_{n=1}^{N-1} \left[A_u(u_n)u_n - A(u_n)\right]\Big|_0^t = \varepsilon \int_0^t \left[\sum_{n=0}^{N-1} \left(\frac{u_{n+1} - u_n}{h}\right)^2\right] dt.$$

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Note: Figure translations are in progress. See original paper for figures.