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Abstract

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LIPSCHITZ SPACES *

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In the paper ([1]) V. A. Efremovich pointed out the necessity of studying properties invariant with respect to transformations satisfying the Lipschitz condition. Just as this has been done for continuity and uniform continuity, an axiomatic approach is also necessary here. The problem of axiomatizing the Lipschitz condition was posed to the author by V. A. Efremovich. In the present note an axiomatics is proposed for spaces for mappings of which it makes sense to speak of the fulfillment of the Lipschitz condition locally (a mapping $f : R \to R'$ satisfies the Lipschitz condition locally if there exist $\varepsilon > 0$ and $L > 0$ such that from $\rho < \varepsilon$ it follows that $\rho' \leq L\rho$, where ρ is the distance between points in the metric space R, and ρ' is the distance between their images in R'). The axiomatization is based on the following theorem: in order that a mapping $f : R \to R'$ satisfy the Lipschitz condition locally, it is necessary and sufficient that for every $\varepsilon > 0$ there exist a $\delta > 0$ such that whenever $\rho(a_1, b_1) + \dots + \rho(a_n, b_n) < \delta$, where n is an arbitrary number and $a_k, b_k \in R$ $(k = 1, 2, \dots, n)$, then $\rho'(f(a_1), f(b_1)) + \dots + \rho'(f(a_n), f(b_n)) < \varepsilon$.

1. Let E be an arbitrary set, $a, b \in E$. The set $p = \{a, b\}$ will be called a pair and will be denoted simply by ab; moreover, we shall not exclude the case when $a = b$ and the pair turns into a one-point set (a degenerate pair). A nonempty finite system (not necessarily of distinct) pairs will be called a class over E. For our purposes it is more convenient to regard a class as a function τ, defined on the set of all pairs and different from zero only on a nonempty finite set, where it assumes natural values. We shall agree to call a pair p, for which $\tau(p) \neq 0$, simply an element of the class τ, and the value $\tau(p)$ the multiplicity of the pair p in the class τ. A class τ consisting of one element of multiplicity 1 will be called simple and will be identified with the pair p for which $\tau(p) \neq 0$. The set of all classes will be denoted by $K(E)$. A class all of whose elements are degenerate pairs will be called degenerate. For a degenerate τ we shall agree to write $\tau = 0$. The class τ' will be called a subclass of the class τ $(\tau' \subset \tau)$ if $\tau'(p) \leq \tau(p)$. We shall say that τ' is contained in τ. The sum $\tau_1 + \tau_2$ of classes and the product $n\tau$ of a class by a natural number are defined as operations on functions. The equality $\tau = \tau_1 + \dots + \tau_n$ will be called a decomposition of the class τ. Every class can be decomposed into a sum of simple classes. The sum $T_1 + T_2$ of sets $T_1, T_2 \subset K(E)$ is defined as the set of all sums $\tau_1 + \tau_2$,

where $\tau_1 \in T_1$, $\tau_2 \in T_2$.

Let us define an order on the set $K(E)$. If $\tau' = ab$, $a \neq b$, and $\tau = aa_1 + a_1a_2 + \dots + a_{nb}$, then we shall say that the class τ' is inscribed in the class τ ($\tau' < \tau$). For an arbitrary τ' containing no degenerate pairs, take its decomposition into a sum of simple classes $\tau' = \tau'_1 + \dots + \tau'_k$. If τ contains the sum of such classes $\tau_1, \dots, \tau_k$ that $\tau'_i < \tau_i$ ($i = 1, 2, \dots, k$), then we put $\tau' < \tau$. Finally, if τ'' is a degenerate class, then we assume that from $\tau' < \tau$ it follows that $\tau' + \tau'' < \tau$, and $\tau'' < \tau_1$ for any τ_1. The order relation introduced is transitive.

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A set $T \subset K(E)$ will be called a domain if from $\tau \in T$, $\tau' < \tau$ it follows that $\tau' \in T$.

Definition. Suppose that on $K(E)$ there is given a filter $\mathfrak{F}$ satisfying the following requirements.

$\mathfrak{F}$1. $\mathfrak{F}$ has a base consisting of domains.

$\mathfrak{F}$2. For every $F \in \mathfrak{F}$ there exists an $F_1 \in \mathfrak{F}$ such that $F_1 + F_2 \in F$.

In this case we shall say that on E a **Lipschitz structure** $\mathfrak{F}$ is defined, and E itself will be called a **Lipschitz space**.

Let us formulate two separation axioms.

O1. The intersection of all $F \in \mathfrak{F}$ consists of all expressed classes.

O2. There exists an $F_0 \in \mathfrak{F}$ such that, for every $\tau \in F_0$, $\tau \neq 0$, for some n we have $n\tau \notin F_0$.

O2 implies O1, but the converse is false. When O1 is satisfied, the structure is called **separated**, and when O2 is satisfied–**strongly separated**.

In the case where E is a metric space, define the diameter of a class by $d\tau' = \sum \tau(p)\,dp$. The filter $\mathfrak{F}$, by definition, is generated by the base $\{F_n\}$, where F_n is the set of all classes whose diameters do not exceed $\frac{1}{n}$.

Let $f : E \to E'$, $\tau \in K(E)$. Define $f(\tau)$ by the formula $f(\tau)(p') = \sum \tau(p)$, where $p' \subset E'$, and the summation extends over all $p \subset E$ for which $f(p) = p'$. If $T \subset K(E)$, then $f(T)$ is the set of all $f(\tau)$, where $\tau \in T$.

Definition. Suppose that on E and E' the structures $\mathfrak{F}$ and $\mathfrak{F}'$, respectively, are defined. A mapping $f : E \to E'$ satisfies the Lipschitz condition in the small if the image of $\mathfrak{F}$ majorizes $\mathfrak{F}'$.

In the case of metric spaces this definition coincides with the metric one.

The set of all Lipschitz structures defined on E is an ordered set, and every one of its subsets has an upper bound (separated or strongly separated, depending on what structures this subset consists of).

2. Suppose $\mathfrak{F}$ is a Lipschitz structure on E. The filter $\mathfrak{F}$ induces on the set of all simple classes a filter $\mathfrak{F}^1$, which generates on E a uniform structure according to the following rule: γ is a two-point system with small sets if and only if γ has nonempty intersection with each element of $\mathfrak{F}^1$. (For systems with small sets see ([2]).) We shall say that the uniform structure obtained in this way is generated by the Lipschitz structure $\mathfrak{F}$.

Theorem 1. *Every separated uniform structure is generated by at least one separated Lipschitz structure. In the set of Lipschitz structures generating a given uniform structure, there is a maximal element.*

In a Lipschitz space one can define certain notions of classical analysis which have no meaning in uniform spaces: order of smallness, contact of curves, absolute convergence of series.

3. Suppose that on the set $A = \{\alpha\}$ a filter $\Phi = \{U\}$ is given, and that on E a strongly separated Lipschitz structure $\mathfrak{F}$ is given. By a function defined on A we shall mean a function taking natural-number values. A mapping $A \to K(E)$ we agree to call a K-mapping, and, if there is no danger of confusion, simply a mapping.

Define the product of a mapping φ by a function f: $(f\varphi)(\alpha) = f(\alpha)\varphi(\alpha)$. We shall call a mapping φ tending to zero ($\varphi \to 0$), or simply convergent, if the image of Φ majorizes the filter $\mathfrak{F}$, and fully divergent if for some $U \in \Phi$ and $F \in \mathfrak{F}$ we have $\varphi(U) \subset K(E) \setminus F$; finally, we shall call φ essential if there exists a $U \in \Phi$ such that $\varphi(\alpha) \neq 0$ for $\alpha \in U$.

For every essential mapping $\varphi \to 0$ there is a function f such that $f\varphi$ is fully divergent.

For the time being we shall restrict ourselves to the consideration of essential mappings. Let $\varphi \to 0$. Denote by $N^n(\varphi)$ the set of all functions f for which $f^n\varphi$ diverges properly.

Let $\varphi_1, \varphi_2 \to 0$.

Definition. The mapping φ_1 has order of smallness $\geq m/n$ in comparison with φ_2 ($\varphi_1 = O(\varphi_2^{m/n})$, or $\varphi_1^n = O(\varphi_2^m)$), if

$$N^m(\varphi_1) \subset N^n(\varphi_2).$$

Definition. The order of smallness of φ_1 in comparison with φ_2 is greater than m/n ($\varphi_1 = o(\varphi_2^{m/n})$, or $\varphi_1^n = o(\varphi_2^m)$), if for some $f \in N^n(\varphi_2)$ we have

$$f^m\varphi_1 \to 0.$$

It is readily verified that $\varphi_1^n = o(\varphi_2^m)$ implies $\varphi_1^n = O(\varphi_2^m)$.

Definition. The order of smallness of φ_1 in comparison with φ_2 is exactly equal to m/n ($\varphi_1^n \sim \varphi_2^m$), if

$$\varphi_1^n = O(\varphi_2^m), \qquad \varphi_2^m = O(\varphi_1^n),$$

i.e. if

$$N^m(\varphi_1) = N^n(\varphi_2).$$

These definitions are correct and, in the metric case, reduce to the following (where φ may be regarded as a numerical function, replacing $\varphi(\alpha)$ by $d\varphi(\alpha)$): $\varphi_1^n = O(\varphi_2^m)$ means that

$$\limsup \frac{\varphi_1^n}{\varphi_2^m} < \infty,$$

and $\varphi_1^n = o(\varphi_2^m)$ means that

$$\varphi_1^n/\varphi_2^m \to 0.$$

Theorem 2. 1) If $\varphi_1, \varphi_2 = o(\varphi)$, then $\varphi_1 + \varphi_2 = o(\varphi)$. 2) If $\varphi_1, \varphi_2 = O(\varphi)$, then $\varphi_1 + \varphi_2 = O(\varphi)$. 3) If $\varphi_1 = O(\varphi)$, but φ_1 is not equivalent to φ, and $\varphi_2 = o(\varphi)$, then $\varphi_1 + \varphi_2 = O(\varphi)$, but $\varphi_1 + \varphi_2$ is not equivalent to φ.

The concept of order of smallness and its properties extend to the case of arbitrary convergent mappings.

4. Let $\lim \Phi = \alpha_0$. To a mapping $g : A \to E$ we assign the K-mapping that takes a point α to the simple class $x_0 g(\alpha)$, where $x_0 = g(\alpha_0)$. Let $g_1(\alpha_0) = g_2(\alpha_0) = x_0$, and let the K-mappings φ_1, φ_2 correspond to the mappings g_1, g_2. Denote by φ_3 the mapping that takes α to the simple class $g_1(\alpha) g_2(\alpha)$.

Theorem 3. If $\varphi_3 = o(\varphi_1)$, then $\varphi_1 \sim \varphi_2$ (and, consequently, $\varphi_2 = o(\varphi_2)$).

Definition. The mappings g_1 and g_2 are tangent at the point α_0 if

$$\varphi_3 = o(\varphi_1).$$

Tangency of mappings at the point α_0 is a reflexive, symmetric, and transitive relation. The order of tangency is also defined. With the aid of the concept of tangency of mappings one can define tangency of curves and surfaces.

One can define a linear Lipschitz space. The concept of order of smallness makes it possible to give a definition of the differential of a mapping of one linear Lipschitz space into another.

5. A sequence $\tau_1, \tau_2, \ldots$ of classes will be called summable if, for every $F \in \mathfrak{F}$, there exists an n_0 such that for $n > n_0$ we have $\tau_{n+1} + \cdots + \tau_m \in F$. In the case of a metric structure this definition means that the series formed from the diameters of the classes converges.

From the summability of the sequence $\{\tau_n\}$ it follows that $\tau_n \to 0$.

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2. V. Yu. Sandberg, *DAN*, **135**, No. 3, 535 (1960).

Note: Figure translations are in progress. See original paper for figures.