



---

Soviet-era science, translated into English

# S. P. Novikov

1962

SovietRxiv

---

View the original and related papers at <https://sovietrxiv.org/items/ru-196201.69528>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

**Abstract**

**Full Text**

**S. P. Novikov**

## ON THE DIFFEOMORPHISM OF SIMPLY CONNECTED MANIFOLDS

*(Presented by Academician P. S. Aleksandrov, 27 XI 1961)*

The object of our investigation is smooth simply connected manifolds of dimension  $n \geq 5$  with a fixed orientation. Two such manifolds are regarded as identical if there exists a diffeomorphism of one onto the other having degree +1 in the chosen orientations. In what follows, by the term “diffeomorphism” we shall mean only a diffeomorphism of degree +1. Our aim is to describe all manifolds  $\{M_i^n\}$  possessing the following properties:

1.  $\pi_1(M_i^n) = 0$ .
2.  $M_i^n$  is homotopy equivalent to  $M_j^n$ .
3. There exists a homotopy equivalence  $f : M_i^n \rightarrow M_j^n$  of degree +1 such that  $f^*V_N(M_j^n) = V_N(M_i^n)$ , where  $V_N(M_s^n)$  is the normal bundle of the manifold  $M_s^n$ , lying in Euclidean space  $E^{N+n}$  for  $N \geq n + 3$ .

It is known <sup>(4)</sup> that under these conditions there is a diffeomorphism

$$M_i^n \times E^N \approx M_j^n \times E^N.$$

It also follows from the works of Smale <sup>(8)</sup> and Hirsch that

$$M_j^n \times S^{N-1} \approx M_i^n \times S^{N-1};$$

however, the manifolds  $M_i^n$  and  $M_j^n$  themselves may fail to be diffeomorphic <sup>(5)</sup>. In the case when the manifolds  $M_i^n$  are homotopy equivalent to the sphere  $S^n$ , their description has been completely carried out in the remarkable papers <sup>(3,7,8)</sup>. Other interesting examples of manifolds have been considered in papers <sup>(9,11)</sup>.

Choose from the class of manifolds  $\{M_i^n\}$ , possessing properties 1-3, some representative  $M_0^n$ , and consider the Thom space <sup>(10)</sup> of the normal bundle  $V_N(M_0^n)$  in Euclidean space of dimension  $N + n$ , which we shall denote by  $T_N$ : the space  $T_N$  is obtained from the bundle  $V_N(M_0^n)$  of closed balls  $R^N$  by contracting the boundary  $\partial V_N$  to a point. There exists a Thom isomorphism

$$\varphi_N : H_i(M_0^n) \rightarrow H_{N+i}(T_N).$$

Denote by  $[M^n]$  the fundamental cycle of the manifold  $M^n$  in the given orientation. The following is obvious:

**Lemma 1.** The cycle  $\varphi_N([M_0^n]) \in H_{N+n}(T_N)$  is spherical.

Denote by  $A \subset \pi_{N+n}(T_N)$  the set of elements such that

$$H(a) = \varphi_N([M_0^n]),$$

where  $a \in \pi_{N+n}(T_N)$  and  $H$  is the Hurewicz homomorphism. From the theory of transversally regular mappings <sup>(10)</sup> it easily follows:

**Lemma 2.** To every element  $a \in A$  there corresponds a manifold

$$M_a^n \subset E^{N+n}$$

and a mapping

$$f_a : M_a^n \rightarrow M_0^n,$$

having degree +1 and such that

$$f_a^* V_N(M_0^n) = V_N(M_a^n).$$

If two manifolds  $M_{a,1}^n$  and  $M_{a,2}^n$ , and mappings

$$f_{a,1} : M_{a,1}^n \rightarrow M_0^n, \quad f_{a,2} : M_{a,2}^n \rightarrow M_0^n,$$

correspond to one and the same element  $a \in A$ , then there exists a manifold

$$N^{n+1} \subset E^{N+n} \times I(0,1)$$

with boundary

$$\partial N^{n+1} = M_{a,1}^n \cup (-M_{a,2}^n),$$

where

$$M_{a,1}^n \subset E^{N+n} \times 0, \quad M_{a,2}^n \subset E^{N+n} \times 1,$$

and there exists a mapping

$$F : N^{n+1} \rightarrow M_0^n$$

such that

$$F^* V_N(M_0^n) = V_N(N^{n+1}), \quad F|_{M_{a,1}^n} = f_{a,1}, \quad F|_{M_{a,2}^n} = f_{a,2}.$$

Define the subset  $\tilde{A} \subset A$ , consisting of elements  $a \in \tilde{A}$  having as representatives manifolds  $M_a^n \in a$  that are homotopy equivalent to the manifold  $M_0^n$ .

**Lemma 3.** If a mapping  $g : M_1^k \rightarrow M_2^k$  of oriented manifolds has degree  $\pm 1$ , then the mapping  $g^* : H^*(M_2^k, K) \rightarrow H^*(M_1^k, K)$  is an isomorphism onto a direct summand, where  $K$  is any field.

From Lemmas 2 and 3 it follows without difficulty:

**Lemma 4.** If  $\alpha \in \tilde{A}$  and the manifold  $M_\alpha^n \in \alpha$  is homotopy equivalent to the manifold  $M_0^n$ , then the mapping  $f_\alpha : M_\alpha^n \rightarrow M_0^n$  is a homotopy equivalence and  $f_\alpha^* V_N(M_0^n) = V_N(M_\alpha^n)$ . If two manifolds  $M_{\alpha,1}^n \in \alpha \in \tilde{A}$ ,  $M_{\alpha,2}^n \in \alpha \in \tilde{A}$  have this property, then the film  $N^{n+1}$ , constructed in Lemma 3, retracts onto each of its boundaries.

In order to understand what an element of the set  $\tilde{A} \subset \pi_{N+n}(T_N)$  is, we shall specify in the normal bundle  $V_N(M_0^n)$  of the manifold  $M_0^n \subset E^{N+n}$  a fixed structure of an  $SO(N)$ -bundle, which we shall call a normal framing of the manifold  $M_0^n$ . This normal framing of the manifold  $M_0^n$  induces a normal framing on the manifold  $M_\alpha^n \in \alpha \in \tilde{A}$ , carried over by means of the mapping  $f_\alpha : M_\alpha^n \rightarrow M_0^n$  from our initial framing on  $M_0^n$ . If two framed manifolds  $M_{\alpha,1}^n \in \alpha$ ,  $M_{\alpha,2}^n \in \alpha$  are equivalent, then the film  $N^{n+1}$ , indicated in Lemma 2, is also normally framed, and the framing on it induces the prescribed framings on the boundaries.

**Lemma 5.** Let a manifold  $M_\alpha^n \in \alpha \in \tilde{A}$  and a mapping  $f_\alpha : M_\alpha^n \rightarrow M_0^n$  be given. Then the set of homotopy classes of normal framings induced by the mapping  $f_\alpha$  from the given framing on the manifold  $M_0^n$  is in one-to-one correspondence with the elements of the group  $\pi(M_\alpha^n, SO(N))$ .

The group  $\pi(M^n, SO(N))$  is, obviously, a homotopy invariant of the manifold  $M^n$ , and therefore on the set  $\tilde{A}$  there naturally acts the group  $\pi(M_0^n, SO(N))$ .

**Lemma 6.** The orbits of the group  $\pi(M_0^n, SO(N))$ , acting on the set  $\tilde{A}$ , contain the same number of elements (note that the set  $\tilde{A}$  is finite).

It can be shown that the action of the group  $\pi(M_0^n, SO(N))$  on the set  $\tilde{A} \subset \pi_{N+n}(T_N)$  is expressed through the Whitehead homomorphism

$$J : \pi_k(SO(q)) \rightarrow \pi_{k+q}(S^q), \quad q > k + 1,$$

whose structure has been studied in dimensions  $k = 4s - 1$  (see <sup>(12)</sup>); for  $k \neq 0, 1 \pmod{8}$  the group  $\pi_k(SO(q))$ ,  $k \neq 4s - 1$ , vanishes (see <sup>(1)</sup>).

The following two theorems are very important for the justification of our constructions.

**Theorem 1.** If  $n \neq 4k + 2$ , then the equality  $\tilde{A} = A$  holds; if  $n = 4k + 2$ , then either  $\tilde{A} = A$ , or  $\tilde{A}$  contains half as many elements as  $A$ , and this depends only on the dimension  $n$  and does not depend on the manifold  $M_0^n$ .

**Theorem 2.** Suppose that the manifolds  $M_1^n$  and  $M_2^n$  are such that: 1)  $\pi_1(M_1^n) = \pi_1(M_2^n) = 0$ ; 2)  $M_1^n$  is homotopy equivalent to  $M_2^n$ , and there exists a homotopy equivalence  $f : M_1^n \rightarrow M_2^n$  such that  $f^* V_N(M_2^n) = V_N(M_1^n)$ , where  $N \geq n + 3$ ,  $V_N(M^n)$  is the normal bundle in the Euclidean space  $E^{N+n} \supset M^n$ ; 3) there exists a manifold  $N^{n+1}$  with boundary  $\partial N^{n+1} = M_1^n \cup (-M_2^n)$ ; 4) there exists a mapping  $g : N^{n+1} \rightarrow M_2^n$  such that  $g|M_2^n = 1$ ,  $g|M_1^n = f$ .

Under these assumptions there exists a Milnor sphere  $\tilde{S}^n \in \theta^n(\partial\pi)$  such that  $M_1^n = (M_2^n \# \tilde{S}^n)$ .

The proof of these theorems is carried out by the method of killing homotopy groups of the homotopy groups by Morse modifications, applied by Milnor and Kervaire to the case of  $\pi$ -manifolds.

- a) To prove Theorem 1 it is necessary to kill a part of the homotopy groups of the manifold  $M_\alpha^n \in \alpha \in A$ . Namely, we have a mapping  $f_\alpha : M_\alpha^n \rightarrow M_0^n$  such that  $f_\alpha^* V_N(M_0^n) = V_N(M_\alpha^n)$ . Let  $x \in \text{Ker } f_{\alpha*}^{(i)}$ ,  $i < n/2$ , where  $f_{\alpha*} : \pi_i(M_\alpha^n) \rightarrow \pi_i(M_0^n)$ , and let  $i$  be such a number that  $\text{Ker } f_{\alpha*}^{(j)} = 0$ ,  $j < i$ .

It can be shown that a sphere  $S^i \subset M_\alpha^n$ , realizing the element  $x \in \text{Ker } f_{\alpha*}^{(i)}$ , has a trivial normal bundle in the manifold  $M_\alpha^n$ . The Morse modification can be carried out so that the normal bundle of the modified manifold in Euclidean space will be “the same” as that of the original one (the expression is imprecise). The normal framing of the manifold  $M_\alpha^n$  can likewise be transferred to the modified manifold for  $i < n/2$ . Analysis of the possibility of performing the modification and of the possibility of transferring the normal framing to the modified manifold for  $i = [n/2]$  reduces this problem to the corresponding problem for the case  $M_0^n = S^n$ , after which we can apply unpublished results of Milnor-Kervaire. Killing the groups  $\text{Ker } f_{\alpha*}^{(i)}$  for  $i < n/2$ , we obtain what is required, by applying Lemmas 3 and 4.

- b) To prove Theorem 2 it is necessary to kill the groups  $\text{Ker } g_*^{(i)}$ , where  $g_*^{(i)} : \pi_i(N^{n+1}) \rightarrow \pi_i(M_2^n)$  in dimensions  $i \leq (n+1)/2$ . Evidently, from the properties of the film  $N^{n+1}$  there follows the direct decomposition:

$$\pi_i(N^{n+1}) = \pi_i(M_j^n) + \text{Ker } g_*^{(i)}, \quad j = 1, 2;$$

$$H_i(N^{n+1}) = H_i(M_j^n) + \text{Ker } g_*^{(i)}, \quad j = 1, 2;$$

$$H^i(N^{n+1}) = H^i(M_j^n) + \text{Coker } g_{(i)}^*, \quad j = 1, 2.$$

Since  $H_i(N^{n+1}, M_1^n) = H_i(N^{n+1}, M_2^n) = \text{Ker } g_*^{(i)}$ , and from the Poincaré duality law it follows that  $H_i(N^{n+1}, M_1^n) = H^{n+1-i}(N^{n+1}, M_2^n)$ , an intersection matrix  $B = (b_{kl})$ ,  $b_{kl} = Z_k \cdot Z_l$ ,  $Z_k, Z_l \in \text{Ker } g_*^{((n+1)/2)}$ , is therefore defined when  $n$  is odd. It is easy to show that this matrix is unimodular (and has even numbers on the diagonal when  $n+1 = 4k$ ). For  $n = 4k - 1$  we shall call the index of the matrix  $B$  the “relative” index and denote it by  $\tau(N^{n+1}, M_1^n, M_2^n)$ . For  $n = 4k - 3$  only some invariant with values in  $\mathbb{Z}_2$  can arise. The groups  $\text{Ker } g_*^{(i)}$  can be killed for  $i < (n+1)/2$  by Morse modifications. When killing the group  $\text{Ker } g_*^{(i)}$  for  $i = (n+1)/2$ , an obstruction may arise, which is easily reduced

to Milnor's group  $\theta^n(\partial\pi)$  (see <sup>(3, 6, 7)</sup>). If this obstruction vanishes, then the manifolds  $M_1^n$  and  $M_2^n$ , evidently, are  $I$ -equivalent and, according to Smale's theorem <sup>(8)</sup>, diffeomorphic.

We shall construct a set  $\tilde{A} \subset \pi_{N+n}(T_N)$ , on which the group  $\pi(M_0^n, SO(N))$  acts. The elements of the quotient set  $\tilde{A}/\pi(M_0^n, SO(N))$  are not, generally speaking, in one-to-one correspondence with the classes of diffeomorphic mod  $\theta^n(\partial\pi)$  manifolds  $\{M_j^n\}$ , homotopy equivalent to the manifold  $M_0^n$  and having a common normal bundle, since there may exist different homotopy equivalences  $f_{ij} : M_j^n \rightarrow M_0^n$  preserving the normal bundle. On the quotient set  $\tilde{A} = \tilde{A}/\pi(M_0^n, SO(N))$  there also acts the group  $\pi^+(M_0^n)$  of homotopy classes of maps of the manifold  $M_0^n$  to itself that have degree +1 and preserve the normal bundle. To describe the action of the latter group, note that to a map  $f : M_0^n \rightarrow M_0^n$  preserving the normal bundle one can associate a map  $\tilde{T}_N f : T_N \rightarrow T_N$ , leaving the set  $\tilde{A}$  invariant and compatible with the action of the group  $\pi(M_0^n, SO(N))$ . Denote...

denote by  $\pi^+(T_N)$  the group of homotopy classes of homotopy equivalences of the space  $T_N$  onto itself, leaving invariant the set  $\tilde{A}$  and the cohomology class  $\varphi(1)$ ; a mapping  $\tilde{T}_N^+$  is defined:

$$\pi^+(M_0^n) \rightarrow \pi^+(T_N).$$

**Lemma 7.** The action of the group  $\pi^+(M_0^n)$  on the set  $\tilde{A} = \tilde{A}/\pi(M_0^n, SO(N))$  is described by the formula  $x(a) = \tilde{T}_N^+ x(a)$ , where  $x \in \pi^+(M_0^n)$ ,  $a \in \tilde{A}/\pi(M_0^n, SO(N))$ .

From geometric considerations it is also easy to extract the following property of the action of the group  $\pi^+(M_0^n)$ . Denote by  $D_a^+ \in \pi^+(M_0^n)$  the subgroup of this group generated by diffeomorphisms of degree +1 of the manifold  $M_a^n \in a \in \tilde{A}/\pi(M_0^n, SO(N))$ ; then we have:

**Lemma 8.** If an element  $x \in \pi^+(M_0^n)$  belongs to the subgroup  $D_a^+$ , then  $x(a) = a$ .

**Remark.** The group  $D_a^+$  depends essentially on the choice of the element  $a \in \tilde{A}/\pi(M_0^n, SO(N))$ ; however, there is an isomorphism  $D_a^+ = D_b^+$  if  $a = x(b)$ ;  $x \in \pi^+(M_0^n)$ .

We can now formulate our main assertion.

**Theorem.** Let a set  $\{M_j^n\}$  of simply connected manifolds of dimension  $n \geq 5$  be given, having a common normal bundle in Euclidean space  $E^{N+n}$  of dimension  $N+n \geq 2n+3$ . A mapping of this set  $\{M_j^n\} \rightarrow (\tilde{A}/\pi(M_0^n, SO(N)))/\pi^+(M_0^n)$  is defined onto the quotient set of a finite family of elements  $\tilde{A} \subset \pi_{N+n}(T_N)$  by the groups  $\pi(M_0^n, SO(N))$  and  $\pi^+(M_0^n)$ , naturally acting in it. If two manifolds

$M_{i_1}^n$  and  $M_{i_2}^n$  go under this mapping into one and the same element, then there exists a Milnor sphere  $\tilde{S}^n \in \theta^n(\partial\pi)$  such that  $M_{i_1}^n = (M_{i_2}^n \# \tilde{S}^n)$  and conversely.

**Example 1** <sup>(3,7,8)</sup>.  $M_0^n = S^n$ ,  $n \neq 4k + 2$ .  $\pi_{N+n}(T_N) = Z + \pi_{N+n}(S^N)$ ;  $y(u_0) = u_0 + h$ , where  $y \in \pi_N(SO(N))$ ,  $h \in \text{Im } J$ ;  $y\tilde{\pi} = 1$ ;  $\tilde{A}/\pi_n(SO(N)) \simeq \pi_{N+n}(S^N)/\text{Im } J$ . The group  $\pi^+(S^n) = 1$ .

**Example 2.**  $M_0^n = S^n \times S^n$ ,  $n = 2k$ .  $\pi_{N+n}(T_N) = Z + \pi_{N+2n}(S^N) + \pi_{N+2n}(S^{N+n}) + \pi_{N+2n}(S^{N+n})$ ;  $\tilde{A}/\pi(M_0^n, SO(N)) = \{1 + \alpha\}$ , where  $\alpha \in \pi_{N+2n}(S^N)/\text{Im } J + \pi_{N+2n}(S^{N+n})/\text{Im } J + \pi_{N+2n}(S^{N+n})/\text{Im } J$ . The elements of the set  $\tilde{A}/\pi(M_0^n, SO(N))$  are determined by three invariants:  $(\alpha, \beta, \beta')$ —elements of the group  $\pi_{N+i}(S^N)/\text{Im } J$ , where  $i = 2n$  or  $i = n$ . It is also obvious that  $\pi^+(M_0^n) = Z_2 + Z_2$ . One can choose such elements  $x, y \in \pi^+(M_0^n)$  that  $x(\alpha, \beta, \beta') = (\alpha, -\beta, -\beta')$ ,  $y(\alpha, \beta, \beta') = (\alpha, \beta', \beta)$ .

**Example 3.** Let  $M_0^8$  be such that  $n = 8$ ,  $\pi_i(M_0^8) = 0$ ,  $i < 4$ ,  $\pi_4(M_0^8) = Z$ . Then we have:  $\pi^+(M_0^8) = 1$ ;  $\pi(M_0^8, SO(N)) = Z_2$ ;  $\pi_{N+8}(T_N) = Z + Z_2 + Z_2$ ;  $\tilde{A} = \{1 + \alpha\}$ ,  $\alpha \in Z_2 + Z_2 = \pi_{N+8}(S^N)$ . Thus the set  $\tilde{A}/\pi(M_0^8, SO(N))$  contains two elements  $\{1 + \alpha_1\}, \{1 + \alpha_2\}$ .

The author expresses deep gratitude to Professors J. Milnor, S. Smale, and F. Hirzebruch for valuable advice and information.

Steklov Mathematical Institute  
Academy of Sciences of the USSR

Received  
9 XI 1961

## CITED LITERATURE

- <sup>1</sup> R. Bott, Ann. of Math., **70**, 313 (1959). <sup>2</sup> A. Haefliger, Bull. Am. Math. Soc., **67**, 1, 109 (1961). <sup>3</sup> M. Kervaire, Comm. Math. Helv., **34**, 4, 257 (1960). <sup>4</sup> B. Mazur, Bull. Am. Math. Soc., **67**, 4, 377 (1961). <sup>5</sup> I. Milnor, Ann. of Math., **64**, 399 (1956). <sup>6</sup> I. Milnor, Am. J. Math., **81**, 962 (1959). <sup>7</sup> I. Milnor, Bull. Soc. Math. France, **87**, 439 (1959). <sup>8</sup> St. Smale, Bull. Am. Math. Soc., **66**, 5, 373 (1960). <sup>9</sup> St. Smale, Ann. of Math., Preprint. <sup>10</sup> R. Thom, Comm. Math. Helv., **28**, 17 (1954). <sup>11</sup> G. T. G. Wall, Ann. of Math., Preprint. <sup>12</sup> J. Milnor, M. Kervaire, Proc. Intern. Congr. Math., 1958, 1960, p. 454.

*Note: Figure translations are in progress. See original paper for figures.*

*Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.*