D. A. Gudkov

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Abstract

Full Text

D. A. Gudkov

Bifurcations of Simple Double Points and Cusps of Real Plane Algebraic Curves

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Mathematics

1. We shall adopt the following notation and terminology: $(x : y : z)$ are the coordinates of a point in the complex projective plane ${}^*R^2$; R^2 is the real projective plane; $R^2 \subset {}^*R^2$; ${}^*E^2$ is the affine complex plane obtained from ${}^*R^2$ by setting $z = 1$; E^2 is the real affine plane belonging to ${}^*E^2$. In ${}^*E^2$ a unitary metric is fixed and, consequently, in E^2 a Euclidean one. $\rho^*(a^0)$ denotes the ρ-neighborhood of a point $a^0 \in {}^*E^2$ in ${}^*E^2$, and $\rho(a^0)$ the ρ-neighborhood of a point $a^0 \in E^2$ in E^2.

It is known that to every real algebraic curve of order m, given by the equation

$$H(x, y, z) \equiv \sum_{\alpha+\beta+\gamma=m} A_{\alpha\beta} x^{\alpha} y^{\beta} z^{\gamma} = 0 \tag{1}$$

(where $A_{\alpha\beta}$ is a nonzero system of real numbers; α, β, γ are nonnegative integers), there corresponds one-to-one the ratio of the coefficients of this curve. The set of such ratios for all real curves of order m is the real projective space

$$R^N \left(N = \frac{m(m+3)}{2} \right).$$

The terms "point" and "curve" in R^N will be taken in quotation marks, in distinction from the same terms in ${}^*R^2$. We shall also call the real curve H a "point" H in the space R^N. Let $E^N_{\alpha_0\beta_0}$ denote the real affine space obtained from R^N if the coefficient $A_{\alpha_0\beta_0}$ is set equal to one. In $E^N_{\alpha_0\beta_0}$ we introduce a Euclidean metric. $S(F, \varepsilon)$ denotes the ε-neighborhood of a point $F \in E^N_{\alpha_0\beta_0}$ in $E^N_{\alpha_0\beta_0}$.

A set Q^n of "points" in some n-dimensional Euclidean space $E^n(\tau_1, \tau_2, \dots, \tau_n)$, defined by the conditions

$$|\tau_j| \le v \quad (j = 1, 2, \dots, n), \tag{2}$$

where $v > 0$ is a constant, is called an n-dimensional element. A subset in Q^n defined by the inequality $\tau_j > 0$ ($\tau_i < 0$) for some j will be called a semielement and denoted by ${}^+Q^n$ (${}^-Q^n$).

A set of "points" in $E^N_{\alpha_0\beta_0}$ homeomorphic to Q^n will be denoted by P^n and will be called an n-dimensional element in R^N, while a set homeomorphic to ${}^+Q^n$ (${}^-Q^n$) will be denoted by ${}^+P^n$ (${}^-P^n$) and will be called a semielement in R^N. If the "point" F is an interior "point" for P^n, we shall write $P^n(F)$. If $P^n(F)$ is given in such a way that all coefficients $A_{\alpha\beta}$ are expressed in terms of n of them by power series with real coefficients convergent in $P^n(E)$, then $P^n(F)$ will be called an analytic element.

2. Let

$$F \equiv \sum_{\alpha+\beta+\gamma=m} A^0_{\alpha\beta} x^\alpha y^\beta z^\gamma = 0$$

be a given curve of order m, having the following singular points:

1°. δ' simple real double points b_s ($s = 1, 2, \dots, \delta'$).

2°. δ'' pairs of imaginary conjugate simple double points a_ν and $\overline{a}_\nu$ ($\nu = 1, 2, \dots, \delta''$).

In all, there are $\delta' + 2\delta'' = \delta$ simple double points.

3°. k' real cusps d_μ ($\mu = 1, 2, \dots, k'$).

4°. k'' pairs of imaginary conjugate cusps e_σ and $\overline{e_\sigma}$ ($\sigma = 1, 2, \dots, k''$).

Altogether $k' + 2k'' = k$ cusps. F has no other singular points in ${}^*R^2$.

Theorem 1*. *There exist such ${}^*E^2$ and $E^N_{\alpha_0\beta_0}$ that all singular points of the curve F lie in ${}^*E^2$, $F \in E^N_{\alpha_0\beta_0}$, and for every sufficiently small $\rho > 0$ there exists an $\varepsilon > 0$ such that:*

1°. Every curve $H \in S(F, \varepsilon) \subset E^N_{\alpha_0\beta_0}$ can have singular points only in the ρ^*-neighborhoods of the singular points of the curve F.

2°. Every curve $H \in S(F, \varepsilon)$ can have in $\rho^*(b_s)$ at most one singular point, which can only be an ordinary real double point. There exists an element ** $P^N_{b_s}(F) \in S(F, \varepsilon)$ such that the set of curves $H \in P^N_{b_s}$ having in $\rho(b_s)$ a singular point is an analytic element $P^{(N-1)}_{b_s}(F)$, dividing the element $P^N_{b_s}$ into two semi-elements ${}^+P^N_{b_s}$ and ${}^-P^N_{b_s}$. The tangent hyperplane to $P^{(N-1)}_{b_s}$ at the "point" F has equation

$$H(b_s) = 0. \tag{3}$$

3°. Every curve $H \in S(F, \varepsilon)$ can have in $\rho^*(a_\nu)$ at most one singular point, which can only be an ordinary imaginary double point, and in this case the curve H has in $\rho^*(\overline{a_\nu})$ the imaginary conjugate ordinary double point. There exists an element $P^N_{a_\nu}(F) \in S(F,\varepsilon)$ in which the curves H having a pair of imaginary conjugate singular points in $\rho^*(a_\nu)$ and $\rho^*(\overline{a_\nu})$ form an analytic element $P^{(N-2)}_{a_\nu}(F)$ with tangent plane at the "point" F determined by the equations

$$\frac{1}{2}\left[H(a_\nu) + H(\overline{a_\nu})\right] = 0, \qquad \frac{1}{2i}\left[H(a_\nu) - H(\overline{a_\nu})\right] = 0. \tag{4}$$

4°. Every curve $H \in S(F,\varepsilon)$ can have in $\rho^*(d_\mu)$ at most one singular point, which can only be real and either an ordinary double point or a cusp. There exists an element $P^N_{d_\mu}(F)$ in which there is an element $P^{(N-1)}_{d_\mu}(F)$ consisting of the curves H having a singular point in $\rho(d_\mu)$. The curves H in $P^N_{d_\mu}$ that have a cusp in $\rho(d_\mu)$ form an analytic element $P^{(N-2)}_{d_\mu}(F)$, dividing $P^{(N-1)}_{d_\mu}$ into two semi-elements: ${}^{+}P^{(N-1)}_{d_\mu}$ (a curve $H \in {}^{+}P^{(N-1)}_{d_\mu}$ has an isolated singular point in $\rho(d_\mu)$), and ${}^{-}P^{(N-1)}_{d_\mu}$ (a curve $H \in {}^{-}P^{(N-1)}_{d_\mu}$ has a node in $\rho(d_\mu)$). The tangent plane to $P^{(N-2)}_{d_\mu}$ at the "point" F has equations

$$H(d_\mu) = 0, \qquad K_\mu(A_{\alpha\beta}, d_\mu) \equiv F_{yy}(d_\mu)H_x(d_\mu) - F_{xy}(d_\mu)H_y(d_\mu) = 0. \tag{5}$$

5°. Every curve $H \in S(F,\varepsilon)$ can have in $\rho^*(e_\sigma)$ at most one singular point, which can only be imaginary and either an ordinary double point or a cusp. Together with the singular point in $\rho^*(e_\sigma)$, the curve H also has the imaginary conjugate singular point of the same type in $\rho^*(\overline{e_\sigma})$. There exists an element $P^N_{e_\sigma}(F) \in S(F,\varepsilon)$, in which there is an element $P^{(N-2)}_{e_\sigma}(F)$, consisting of the curves H having a singular point in $\rho^*(e_\sigma)$. The curves H in $P^N_{e_\sigma}$, having a cusp at $P^*(e_\sigma)$, form an analytic element $P^{(N-4)}_{e_\sigma}(F) \subset P^{(N-2)}_{e_\sigma}$ with tangent plane defined by the equations

$$\frac{1}{2}\left[H(e_\sigma) + H(\bar{e}_\sigma)\right] = 0, \qquad \frac{1}{2i}\left[H(e_\sigma) - H(\bar{e}_\sigma)\right] = 0,$$

$$\frac{1}{2}\left[K_\sigma(A_{\alpha\beta}, e_\sigma) + K_\sigma(A_{\alpha\beta}, \bar{e}_\sigma)\right] = 0, \qquad \frac{1}{2i}\left[K_\sigma(A_{\alpha\beta}, e_\sigma) - K_\sigma(A_{\alpha\beta}, \bar{e}_\sigma)\right] = 0. \tag{6}$$

* Theorem 1 is a refinement and generalization (to curves with cusps) of Theorem 1 of (*). When writing (4) we were not acquainted with work (2), and in Theorems 1, 2, 5, 6 (4) repeated some results of Brusotti, as Golafassi pointed out to us (5). We take this opportunity to express our gratitude to Golafassi.

** The sign b_s on the element $P^N(F)$ will be used to distinguish elements related to different singular points of the curve F.

Lemma 1. Let the curve F decompose into components F_h $(h = 1, 2, \dots, q)$. Let $m_h \geq 1$ be the order and k_h the number of cusps of the component F_h. If the conditions

$$k_h \leq 3m_h - 1 \qquad (h = 1, 2, \dots, q), \tag{7}$$

are satisfied, then the system of $(\delta + 2k)$ linear equations (with respect to all coefficients $A_{\alpha\beta}$ of the curve H), consisting of equations (3), (4), (5), and (6) for all singular points of the curve F, is linearly independent.*

Theorem 2. If conditions (7) are satisfied for the curve F, then in the space $E_{\alpha_0\beta_0}$ defined in Theorem 1 there exists such an element $P^N(F)$, containing in each of the elements: $P^N_{b_s}$ $(s = 1, 2, \dots, \delta')$, $P^N_{a_\nu}$ $(\nu = 1, 2, \dots, \delta'')$, $P^N_{d_\mu}$ $(\mu = 1, 2, \dots, k')$, $P^N_{e_\sigma}$ $(\sigma = 1, 2, \dots, k)$, that:

1°. The curves $H \in P^N$, having in the P^*-neighborhoods** of the simple double points of the curve F simple double points, and in the P^*-neighborhoods of the cusps of the curve F cusps, form an analytic element $P^{N-(\delta+2k)}(F)$, having at the "point" F a tangent plane defined by the system of $(\delta + 2k)$ equations: (3), (4), (5), and (6) for all singular points of the curve F.

2°. There exists a curve $G \in P^N$ and a continuous "curve" $FG \subset P^N$ (the "curve" FG joins the "points" F and G) such that:

a) for each of the points b_s $(s = 1, 2, \dots, \delta')$ (independently of one another), at our choice either $FG \subset P^{N-1}_{b_s}$, or $FG \subset {}^+P^N_{b_s}$ (except for the "point" F), or $FG \subset {}^-P^N_{b_s}$ (except for the "point" F);

b) for each of the points a_ν $(\nu = 1, 2, \dots, \delta'')$ (independently of one another), at our choice either $FG \subset P^{N-2}_{a_\nu}$, or $FG \subset P^N_{a_\nu} \setminus P^{N-2}_{a_\nu}$ (except for the "point" F);

c) for each of the points d_μ $(\mu = 1, 2, \dots, k')$ (independently of one another), at our choice either $FG \subset P^{(N-2)}_{d_\mu}$, or $FG \subset {}^+P^{(N-1)}_{d_\mu}$ (except for the "point" F), or $FG \subset {}^-P^{(N-1)}_{d_\mu}$ (except for the "point" F);

d) for each of the points e_σ $(\sigma = 1, 2, \dots, k'')$ (independently of one another), at our choice either $FG \subset P^{(N-4)}_{e_\sigma}$, or $FG \subset P^{(N-2)}_{e_\sigma} \setminus P^{(N-4)}_{e_\sigma}$ (except for the "point" F).

Moreover, if $H \in FG$, then the coordinates of all singular points of the curve H are continuous functions of the coefficients of the curve H.***

* Brusotti proved (2) a special case of this lemma, when $k = 0$.

** All singular points of the curve F lie in the plane ${}^*E^2$, defined in Theorem 1.

*** If assertion 2° of Theorem 2 holds, then we shall say that all singular points of the curve F can be bifurcated by arbitrarily small additions independently of one another.

Lefschetz stated (1) the theorem:

If an irreducible curve F has δ simple double points, then each double point imposes one condition on the coefficients of the curve F.

In fact, Lefschetz proved this assertion only for unicursal curves having only simple double points.

In the same work Lefschetz accepted without proof the postulate on singularities:

If a curve F is irreducible and has δ simple double points and k cusps (and has no other singularities), then each simple double point imposes one condition, and each cusp two conditions, on the coefficients of the curve F.

Coolidge proves (3) the validity of the "postulate on singularities" if $k \leq 3m-1$. Like Lefschetz, Coolidge proves his theorems for complex irreducible curves and does not address the question of the independence of bifurcations of singular points.

Brusotti proved (2) a special case of Theorem 2, when $k = 0$. Our Theorem 2 is a generalization of the indicated theorem of Brusotti to curves with cusps.

Theorem 3. Let the curve F decompose into components F_h $(h = 1, 2, \dots, q)$ of orders $m_h \geqslant 1$ and genera p_h. If the conditions

$$p_h \leqslant \frac{m_h + 4}{2} \qquad (h = 1, 2, \dots, q), \tag{8}$$

are satisfied, then the assertion of Theorem 2 holds.

Research Physico-Technical Institute
of Gorky State University
named after N. I. Lobachevsky

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Note: Figure translations are in progress. See original paper for figures.