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Academician A. N. KOLMOGOROV, E. F. MISHCHENKO, and  
Academician L. S. PONTRYAGIN

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**Abstract**

**Full Text**

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### MATHEMATICS

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## ON A PROBABILISTIC PROBLEM OF OPTIMAL CONTROL

Let  $p(\sigma, x, \tau, y)$  be the probability density of a Markov process in  $n$ -dimensional Euclidean space  $R_n$  ( $n \geq 3$ ), subject to the Kolmogorov equation <sup>(1)</sup>

$$\frac{\partial p}{\partial \sigma} + \sum_{i,j=1}^n a^{ij}(\sigma, x) \frac{\partial^2 p}{\partial x^i \partial x^j} + \sum_{i=1}^n b^i(\sigma, x) \frac{\partial p}{\partial x^i} = 0. \quad (1)$$

Let a second point  $z$  move in the same space  $R_n$  deterministically according to the law  $z = z(t)$ . Together with  $z$ , its neighborhood moves, bounded by the closed surface  $\Sigma_t = z(t) + \varepsilon \Sigma$ , similar with a small similarity coefficient  $\varepsilon$  to the fixed surface  $\Sigma$  (in what follows, for simplicity, we shall take  $\Sigma$  to be the sphere of unit radius). It is required to determine the probability  $\psi(\sigma, x, \tau)$  that a random point, whose transition density is subject to equation (1), will intersect the surface  $\Sigma_t$  in the time interval  $\sigma \leq t \leq \tau$ .

This problem was solved by E. F. Mishchenko and L. S. Pontryagin in <sup>(2)</sup> in connection with the needs of optimal control. However, the approximate formula for the probability  $\psi$  obtained in that work turned out to be cumbersome and of little use for further application.

A. N. Kolmogorov, having become acquainted with <sup>(2)</sup>, proposed, on probabilistic grounds, another, considerably simpler expression for the approximation of E. F. Mishchenko and L. S. Pontryagin. However, he did not give a proof.

In the present note we give Kolmogorov's formula and its proof, proposed by E. F. Mishchenko and L. S. Pontryagin. This proof is based on the constructions given in <sup>(2)</sup>.

It is known (cf. <sup>(2)</sup>) that the desired probability  $\psi(\sigma, x, \tau)$  is a solution of equation (1) and satisfies the conditions

$$\psi(\tau, x, \tau) = 0, \quad \psi(\sigma, x, \tau)|_{\Sigma_\sigma} = 1. \quad (2)$$

A. N. Kolmogorov proposed the following formula for the principal part  $K(\sigma, x, \tau, \varepsilon)$  of the probability  $\psi$ :

$$K(\sigma, x, \tau, \varepsilon) = \varepsilon^{n-2} \int_\sigma^\tau p(\sigma, x, s, z(s)) \beta(s) ds, \quad (3)$$

where

$$\beta(s) = \int_{A_s \Sigma} \frac{\partial w(s, \xi)}{\partial n} d\Sigma; \quad (4)$$

$A_s$  is the linear transformation  $\xi = A_s \bar{\xi}$  which brings the differential form

$$\sum a^{ij}(s, z(s)) \frac{\partial^2}{\partial \xi^i \partial \xi^j}$$

to the form

$$\sum_{k=1}^n \frac{\partial^2}{\partial \xi^{k^2}},$$

and  $w(s, \xi)$  is a harmonic function satisfying the conditions:

$$w(s, \xi) = 1 \quad \text{for } \xi \in A_s \Sigma; \quad w(s, \xi) \rightarrow 0 \quad \text{as } |\xi| \rightarrow \infty.$$

It is verified directly that the function  $K(\sigma, x, \tau, \varepsilon)$ , defined by formula (3), satisfies equation (1) outside the point  $z(\sigma)$ .

We shall prove that on a certain specially chosen small ellipsoid with center at the point  $z(\sigma)$ , the function  $K(\sigma, x, \tau, \varepsilon)$  and the function  $\Psi(\sigma, x, \tau, \varepsilon)$ , constructed in [2] and constituting the principal part of the probability  $\psi(\sigma, x, \tau)$ , differ “inessentially,” i.e., coincide up to  $o(\varepsilon)$  for  $\tau - \sigma \geq \varepsilon$  and differ only by  $O(1)$  for  $\tau - \sigma < \varepsilon$ . Hence, by virtue of Lemma 3 of [2], it follows that

$$K(\sigma, x, \tau, \varepsilon) = \Psi(\sigma, x, \tau, \varepsilon).$$

For the proof, introduce in the space  $(z, t)$  new coordinates by the formulas  $z = \xi + z(t)$ ,  $\sigma \leq t \leq s$ , so that  $x = \xi + z(\sigma)$ ,  $y = \eta + z(s)$ . Next put  $\xi = A_\sigma \bar{\xi}$ . Under this change of coordinates, the function  $K(\sigma, x, \tau, \varepsilon)$  becomes a function

$Q(\sigma, \xi, \tau, \varepsilon)$ , while the function  $\Psi(\sigma, x, \tau, \varepsilon)$  becomes a function  $\Phi(\sigma, \xi, \tau, \varepsilon)$ . Obviously,

$$Q(\sigma, \xi, \tau, \varepsilon) = \varepsilon^{n-2} \int_{\sigma}^{\tau} q(\sigma, \xi, s, 0) \beta(s) ds, \quad (5)$$

where

$$q(\sigma, \xi, s, \eta) = p(\sigma, A_{\sigma}^{-1}\xi + z(\sigma), s, A_s^{-1}\eta + z(s)). \quad (6)$$

The function  $q(\sigma, x, \tau, \eta)$  is the fundamental solution of the parabolic equation obtained from equation (1) upon passing to the coordinates  $\xi$ .

In [2] it is shown that the function  $\Phi(\sigma, \xi, \tau, \varepsilon)$ , for  $|\xi| = \varepsilon$ , differs only “inessentially” from the quantity  $\alpha(\sigma)$ , which arises as follows. We shall solve the Dirichlet problem for the equation  $\Delta w = 0$  under the conditions

$$w(\sigma, \xi)|_{H_{\sigma}} = 1, \quad w(\sigma, \xi) \rightarrow 0 \quad \text{as } |\xi| \rightarrow \infty.$$

Here  $H_{\sigma}$  is the ellipsoid obtained from the sphere  $\varepsilon\Sigma$  by the transformation  $A_{\sigma}$ . As is known, the function  $w(\sigma, \xi)$  can be represented in the form

$$w(\sigma, \xi) = \frac{\varepsilon^{n-2}\alpha(\sigma)}{r^{n-2}(\xi)} + \Pi(\sigma, \xi, \varepsilon), \quad (7)$$

where  $\Pi(\sigma, \xi, \varepsilon)$  is the double-layer potential produced by the ellipsoid  $H_{\sigma}$  at the point  $\xi$ .

It is not difficult to establish the relation between  $\alpha(\sigma)$  and  $\beta(\sigma)$ , which occurs in formula (4). Indeed, if one takes into account that the integral of the normal derivative over the surface  $H_{\sigma}$  of the double-layer potential  $\Pi(\sigma, \xi, \varepsilon)$  is equal to zero, then, differentiating the right- and left-hand sides of relation (7) along the normal to  $H_{\sigma}$  and then taking the integral over  $H_{\sigma}$ , we find that

$$\beta(\sigma) = \frac{4\pi^{n/2}}{\Gamma(n/2 - 1)} \alpha(\sigma), \quad (8)$$

where  $\Gamma$  is the gamma function.

Let us show that the Kolmogorov function  $Q(\sigma, \xi, \tau, \varepsilon)$ , defined by formula (5), also differs only “inessentially” from  $\alpha(\sigma)$  when  $|\xi| = \varepsilon$ .

Using the arguments and estimates of [2], one can first of all show that

$$Q(\sigma, \xi, \tau, \varepsilon)|_{|\xi|=\varepsilon} = \varepsilon^{n-2} \int_{\sigma}^{\tau} \gamma(\sigma, \xi, s, 0)|_{|\xi|=\varepsilon} \beta(s) ds + o(1), \quad (9)$$

where  $\gamma(\sigma, \xi, s, \eta)$  is the Green function of the heat equation:

$$\gamma(\sigma, \xi, s, \eta) = \frac{1}{[4\pi(s - \sigma)]^{n/2}} e^{-|\xi - \eta|^2/4(s - \sigma)}. \quad (10)$$

We compute the value of the integral  $\varepsilon^{n-2} \int_{\sigma}^{\tau} \gamma(\sigma, \xi, s, 0) \beta(s) ds$  for  $|\xi| = \varepsilon$ .

We have

$$\begin{aligned} \varepsilon^{n-2} \int_{\sigma}^{\tau} \gamma(\sigma, \xi, s, 0) \Big|_{|\xi|=\varepsilon} \beta(s) ds &= \varepsilon^{n-2} \int_{\sigma}^{\tau} \gamma(\sigma, \xi, s, 0) \Big|_{|\xi|=\varepsilon} [\beta(\sigma)\beta(s) - \beta(\sigma)] ds \\ &= \frac{\beta(\sigma)\varepsilon^{n-2}}{(4\pi)^{n/2}} \int_{\sigma}^{\tau} \frac{1}{(s-\sigma)^{n/2}} e^{-\varepsilon^2/4(s-\sigma)} ds \\ &\quad + \varepsilon^{n-2} \int_{\sigma}^{\tau} \gamma(\sigma, \xi, s, 0) \Big|_{|\xi|=\varepsilon} [\beta(s) - \beta(\sigma)] ds. \end{aligned} \tag{11}$$

Put  $s - \sigma = \varepsilon^2 t$ . Then

$$\frac{\beta(\sigma)}{(4\pi)^{n/2}} \int_{\sigma}^{\tau} \frac{1}{(s-\sigma)^{n/2}} e^{-\varepsilon^2/4(s-\sigma)} ds = \frac{\beta(\sigma)}{(4\pi)^{n/2}} \int_0^{\infty} \frac{1}{t^{n/2}} e^{-1/4t} dt + \omega(\varepsilon, \sigma, \tau),$$

where  $\omega(\varepsilon, \sigma, \tau)$  is bounded for  $\tau - \sigma \leq \varepsilon$  and has magnitude of order  $o(1)$  for  $\tau - \sigma > \varepsilon$ . Making the substitution  $x = 1/4t$ , we obtain

$$\frac{\beta(\sigma)}{(4\pi)^{n/2}} \int_0^{\infty} \frac{1}{t^{n/2}} e^{-1/4t} dt = \frac{\beta(\sigma)}{4\pi^{n/2}} \Gamma\left(\frac{n}{2} - 1\right) = \alpha(\sigma). \tag{12}$$

Further, it is easy to verify that

$$\varepsilon^{n-2} \int_{\sigma}^{\tau} \gamma(\sigma, \xi, s, 0) \Big|_{|\xi|=\varepsilon} [\beta(s) - \beta(\sigma)] ds = o(1). \tag{13}$$

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## CITED LITERATURE

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*Note: Figure translations are in progress. See original paper for figures.*

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