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## Abstract

## Full Text

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## Uniform Approximation of Generalized Analytic Functions

*(Presented by Academician I. N. Vekua, January 18, 1962)*

Generalized analytic functions ($^1$, $^2$) are the solutions of equations of the form

$$w_{\bar{z}} + A(z)w + B(z)\overline{w} = 0, \tag{1}$$

where $A(z)$ and $B(z) \in L_{p,2}$ $(p > 2)$. In the book ($^2$) (pp. 119-122)* the following theorem is proved.

**Theorem 1.** *If $f(z)$ is a regular solution of equation (1) in a domain $D$, the complement of $\overline{D}$ consists of $k$ domains, and $z_1, \dots, z_k$ are arbitrary points from these domains, then $f(z)$ can be expanded in a series, uniformly convergent on any closed set $E \subset D$, in rational generalized analytic functions satisfying the same equation and having poles only at the points $z_1, \dots, z_k$.*

In the present work the following theorem on the approximation of generalized analytic functions is proved; from it there follow simply analogues of the theorems of S. N. Mergelyan ($^3$), p. 44, Theorem 4, and of A. G. Vitushkin ($^4$), p. 961, Theorem 4.

**Theorem 2.** *Suppose that every function continuous on the set $E$ and analytic at the interior points of $E$ can be expanded on $E$ in a uniformly convergent series of rational functions having poles only at points $z_1, \dots$ exterior to $E$ (the number of points may be finite or countable). Then every function continuous on $E$ and satisfying equation (1) at the interior points of $E$ can be expanded on the set $E$ in a uniformly convergent series of rational generalized analytic functions satisfying the same equation and having no other singularities except poles at the same points.*

**Lemma.** *If a series $\sum_{n=1}^{\infty} \varphi_n(z)$ of functions continuous on $\overline{E}$ converges uniformly on $E$, then it also converges uniformly on $\overline{E}$.*

For the proof it suffices to apply the Cauchy criterion for uniform convergence of functional series.

**Corollary.** A series of rational functions uniformly convergent on $E$ converges to a function continuous on $\overline{E}$ and analytic at the interior points of $\overline{E}$. In proving Theorem 1 the set $E$ may be regarded as closed. Each domain of the complement of $E$ contains at least one point $z_k$.

**Proof of Theorem 2.**
1. $B(z) \equiv 0$. Equation (1) has the form

$$w_{\bar{z}} + A(z)w = 0; \tag{2}$$

its solutions are connected with analytic functions by the integral representation –

* In the book ($^2$), $A(z)$ and $B(z)$ are assumed to be Hölder continuous. Taking into account the results of the book ($^1$), this restriction can easily be removed.

equation of the first kind ((1), p. 159)

$$w(z) = \Phi(z)e^{-\omega(z)}.$$

If $f(z)$ is continuous on $E$ and satisfies equation (2) at the interior points of $E$, then the function $\Phi(z) = f(z)e^{\omega(z)}$ can, by the condition, be expanded in a series uniformly convergent on $E$,

$$\sum_{n=1}^{\infty} \Phi_n(z),$$

of rational functions with poles at the prescribed points. The series

$$\sum_{n=1}^{\infty} \Phi_n(z)e^{-\omega(z)}$$

will be the desired series for $f(z)$.

2. $B(z) \equiv 0$ outside $E$. A function $f(z)$ satisfying equation (1) satisfies the equation

$$w_{\bar{z}} + \left[A(z) + B(z)\frac{\overline{f(z)}}{f(z)}\right] w = 0. \tag{2'}$$

By what has been proved, there is an expansion

$$f(z) = \sum_{n=1}^{\infty} \varphi_n(z), \tag{3}$$

where $\varphi_n(z)$ and the partial sums $s_n(z)$ satisfy equation (2′) and the equations

$$w_{\bar{z}} + A(z)w + B(z)\overline{w} = \left[\overline{s_n(z)} - s_n(z)\frac{\overline{f(z)}}{f(z)}\right] B(z) = F_n(z), \tag{4}$$

whose right-hand sides are small in $L_{p,2}$. By the absolute continuity of the operator $T$ ((1), p. 62), the sequence of functions

$$w_n^*(z) = (T + RT)F_n(z) \tag{5}$$

converges uniformly to zero as $n \to \infty$. The functions

$$s_n^*(z) = s_n(z) - w_n^*(z) \tag{6}$$

satisfy ((1), p. 169) equation (1) and converge uniformly to $f(z)$ on $E$.

3. To prove Theorem 2 in the general case, we construct a sequence of domains $D_n$ containing $E$, having a finite number of connected complementary domains, each of which contains one of the points $z_k$. We choose the domains $D_n$ so that the functions

$$w_n^{**}(z) = (T_{D_n} + R_{D_n} T_{D_n})F_n(z) \tag{5'}$$

tend to zero as $n \to \infty$. Then the functions

$$s_n^{**}(z) = s_n(z) - w_n^{**}(z)$$

converge uniformly on $E$ to $f(z)$ and satisfy equation (1) in $D_n$. Applying Theorem 1 to them, we obtain the validity of Theorem 2.

**Corollary 1.** If, under the hypotheses of Theorem 1, the function $f(z)$ has zeros at a finite number of interior points of $E$, then the theorem remains valid, only the approximating functions will have poles at the same points. The proof is obvious.

**Corollary 2.** Solutions of the equation

$$w_{\bar{z}} + A(z)w + B(z)\overline{w} = F(z) \tag{4'}$$

can be expanded in a uniformly convergent series of the form

$$f(z) = (T + RT)F(z) + \sum_{n=1}^{\infty} \varphi_n(z), \tag{7}$$

where $\varphi_n(z)$ have poles at the prescribed points and satisfy equation (1), while the partial sums satisfy equation (4). Using expansion (7), one can prove the validity of the assertion converse to Theorem 2.

**Consequence 3.** Since any function $f(z)$ satisfying the equation

$$w_{\bar{z}} = F(z, w, w_z), \tag{8}$$

satisfies an equation of the form (4), it can be expanded in a series in solutions of the latter.

**Consequence 4.** Theorem 2 and the converse assertion are also valid for solutions of the equation

$$u_{\bar{z}} = q(z) u_z, \tag{9}$$

where $|q(z)| \leq q_0 < 1$ and $q_{\bar{z}} \in L_{p,2}$ $(p > 2)$. For the proof it is sufficient to note that the function

$$w(z) = u(z) - \overline{qu}(z)$$

satisfies the equation

$$w_{\bar{z}} - \frac{q_{\bar{z}}\overline{q(z)}}{1 - |q(z)|^2}\, w - \frac{q_{\bar{z}}}{1 - |q(z)|^2}\, \overline{w} = 0. \tag{10}$$

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*Note: Figure translations are in progress. See original paper for figures.*

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