MATHEMATICS

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1962

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Abstract

Full Text

MATHEMATICS

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ON REGULAR FUNCTIONS WITH BOUNDED MEAN MODULUS

(Presented by Academician V. I. Smirnov on 28 V 1962)

Let us denote by H_δ $(\delta > 0)$ the class of functions

$$f(\zeta) = \sum_{n=0}^{\infty} c_n \zeta^n, \tag{1}$$

regular in the disk $|\zeta| < 1$ and satisfying, for $0 < r < 1$, the condition

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^\delta \, d\theta \leq 1. \tag{2}$$

It is easy to see that $H_{\delta_1} \subset H_{\delta_2}$ if $\delta_1 > \delta_2$.

Theorem 1. If $f(z) \in H_\delta$, $\delta > 1$, then for $n = 1, 2, ...$, with $\zeta = z$, $|z| = r < 1$, the estimate holds

$$|f^{(n)}(z)| \leq \frac{n!}{(1-r^2)^{n+1/\delta}} F^{\frac{\delta-1}{\delta}} \left(-\frac{\delta(n-1)+2}{2(\delta-1)}, -\frac{\delta(n-1)+2}{2(\delta-1)}, 1; \, r^2 \right), \tag{3}$$

where $F(\alpha, \beta, \gamma; z)$ is the hypergeometric series. The order of this estimate as $r \to 1$ is sharp.

Proof. We have

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{|\zeta|=1} \frac{f(\zeta)\, d\zeta}{(\zeta - z)^{n+1}} \qquad (n = 1, 2, ...).$$

Using Hölder' s inequality, putting $\frac{1}{\delta} + \frac{1}{\delta_1} = 1$, as in paper ([1]), we shall have

$$|f^{(n)}(z)| \leq \frac{n!}{2\pi} \left\{ \int_{|\zeta|=1} |f(\zeta)|^\delta \, |d\zeta| \right\}^{1/\delta} \left\{ \int_0^{2\pi} \frac{d\theta}{|1 - re^{i\theta}|^{\delta_1(n+1)}} \right\}^{1/\delta_1} =$$

$$= 2\pi F\left(\frac{\delta_1(n+1)}{2}, \frac{\delta_1(n+1)}{2}, 1; r^2\right).$$

Hence, applying the formula

$$F(\alpha,\beta,\gamma;z) = (1-z)^{\gamma-\alpha-\beta} F(\gamma-\alpha, \gamma-\beta, \gamma; z),$$

we obtain estimate (3).

An example of a function

$$f(\zeta) = \left[\frac{1-r^2}{(1-z\zeta)^2}\right]^{1/\delta} \in H_\delta, \qquad r = |z|, \quad |\zeta| < 1, \tag{4}$$

shows that the order of estimate (3) as $r \to 1$ is sharp.

Remark. When $\delta(n-1)/2(\delta-1)$ is equal to m, a positive integer, formula (3) gives

$$|f^{(n)}(z)| \leq \frac{n!}{(1-r^2)^{n+1/\delta}} \left[1 + \binom{m}{1}^2 r^2 + \cdots + \binom{m}{m}^2 r^{2m}\right]^{(\delta-1)/\delta}.$$

This estimate is sharp and was previously found in paper ([2]), p. 301.

Letting $\delta \to \infty$ in (3), we obtain the estimate for the class of bounded functions previously found by Szasz ([3]).

Theorem 2. If $f(\zeta) = c_m\zeta^m + c_{m+1}\zeta^{m+1} + \cdots \in H_1$ $(m = 0, 1, 2, \ldots;\ c_m \neq 0)$, then for $\zeta = z,\ |z| = r < 1$,

$$|f'(z)| \leq \frac{r^{m-1}\left[m + (2-m)r^2\right]}{2(1-r^2)^2} \quad \text{when} \quad |f(z)| \leq \frac{r^m}{2(1-r^2)}, \tag{5}$$

$$|f'(z)| \leq \frac{r^{m-1}\left[m(1-r^2) + 2r^2 + \sqrt{m^2(1-r^2)^2 + 4m(1-r^2)r^2 + 4r^4 + 4r^2}\right]}{2(1-r^2)^2}$$

$$\text{when} \quad |f(z)| \geq \frac{r^m}{2(1-r^2)}. \tag{6}$$

The estimate is sharp for every $|z| = r$, and equality is attained for the function

$$f(\zeta) = \zeta^m \left(\sqrt{\frac{\sqrt{4r^2+k^2}+k}{2\sqrt{4r^2+k^2}}} + \sqrt{\frac{\sqrt{4r^2+k^2}-k}{2\sqrt{4r^2+k^2}}}\, \frac{\zeta - z}{1-\bar z \zeta} \right)^2 \frac{1-r^2}{(1-\bar z\zeta)^2} \in H_1,$$

where $k = m(1-r^2) + 2r^2,\ r = |z|,\ |\zeta| < 1$.

Proof. For a function $a_0 + a_1 z + \cdots \in H_1$, G. M. Goluzin proved ([4], p. 72) that for $n > 2k$

$$|a_n| \le \begin{cases} 1, & \text{when } |a_k| \le 1/2, \\ 2\sqrt{|a_k|(1-|a_k|)}, & \text{when } |a_k| \ge 1/2. \end{cases} \tag{7}$$

Put $\varphi(z) = f(z)/z^m = c_m + c_{m+1} z + \cdots$. If $f(z) \in H_1$, then also $\varphi(z) \in H_1$. Consider the function

$$g(\zeta) = \varphi\left(\frac{\zeta + z}{1+\bar z\zeta}\right) \frac{1-|z|^2}{(1+\bar z\zeta)^2} \in H_1 \quad (|\zeta| < 1).$$

Applying (7) to the function $g(\zeta)$ for $k = 0$ and $n = 1$, we estimate $|f'(z)|$ as a function of $|f(z)|$, whence the estimates (5) and (6) follow.

For $m = 0$, (6) gives the estimate of G. M. Goluzin ([4], p. 34), and for $m = 1$ the estimate of Macintyre-Rogosinski ([2], p. 317).

Theorem 3. If $f(\zeta) \in H_\delta,\ f(\zeta) \neq 0$ for $|\zeta| < 1$, and $0 < \delta \le 1$, then for $\zeta = z,\ |z| = r < 1$,

$$|f'(z)| \le \frac{2}{\delta(2-\delta)} \frac{r(1-\delta) + \sqrt{2\delta - \delta^2 + r^2}}{(1-r^2)^{1+1/\delta}} \left\{ \frac{2 - \delta + r^2 + r\sqrt{2\delta-\delta^2+r^2}}{2(1+r^2)} \right\}^{1/\delta}. \tag{8}$$

The estimate is sharp, and equality is attained for the function

$$f(\zeta) = \left(a_0 + a_1 \frac{\zeta - z}{1-\bar z\zeta}\right)^{2/\delta} \frac{(1-r^2)^{1/\delta}}{(1-\bar z\zeta)^{2/\delta}} \in H_\delta,$$

where

$$a_0 = \sqrt{\frac{2-\delta+r^2+r\sqrt{2\delta-\delta^2+r^2}}{2(1+r^2)}}; \qquad a_1 = \sqrt{\frac{\delta+r^2-r\sqrt{2\delta-\delta^2+r^2}}{2(1+r^2)}}.$$

Proof. The function $f_1(\zeta) = [f(\zeta)]^{\delta/2} \in H_2$. From the inequality

$$|f_1(0)|^2 + |f_1'(0)|^2 \leq 1$$

we find

$$|f'(0)| \leq \frac{2}{\delta}\sqrt{|f(0)|^{2-\delta} - |f(0)|^2}. \tag{9}$$

Applying (9) to the function

$$g(\zeta) = f\left(\frac{\zeta + z}{1 + \bar{z}\zeta}\right)\left[\frac{1-r^2}{(1+\bar{z}\zeta)^2}\right]^{1/2} \in H_\delta, \qquad r = |\zeta| < 1,$$

we obtain

$$|f'(z)| \leq \frac{2}{\delta(1-r^2)}\left\{r|f(z)| + \sqrt{\frac{|f(z)|^{2-\delta}}{1-r^2} - |f(z)|^2}\right\}.$$

Determining the greatest value of the right-hand side as a function of $|f(z)|$ on the interval $[0, (1-r^2)^{-1/\delta}]$, we obtain the estimate (8).

Using the results of G. M. Goluzin (([4]), pp. 34 and 45), one can prove the following two theorems.

Theorem 4. If $f(z) \in H_\delta$ $(\delta > 0)$, then

$$|f(z)| \leq \frac{1}{(1-r^2)^{1/\delta}}, \qquad r = |z| < 1. \tag{10}$$

The estimate is sharp, and equality is attained for the function (4).

This theorem is known ([5]). In what follows we use its result.

Theorem 5. For any integer $n \geq 1$ and $\alpha = e^{2\pi i/n}$, for a function $f(\zeta) \in H_\delta$ $(\delta > 0)$, for $|z| = r < 1$, we have the estimate

$$\sum_{k=1}^{n} |f(\alpha^k z)| \leq \frac{n}{(1-r^2)^{1/\delta}} \qquad \text{for } \delta \geq 1, \tag{11}$$

$$\sum_{k=1}^{n} |f(\alpha^k z)| \leq \left(\frac{n}{1-r^{2n}}\right)^{1/\delta} \qquad \text{for } \delta \leq 1. \tag{12}$$

The estimate (11) is sharp, and equality is attained for the function

$$f(\zeta) = \left[\frac{1-r^{2n}}{(1-z^n\zeta^n)^2}\right]^{1/\delta}, \qquad r = |z|.$$

The estimate (12) is asymptotically sharp, as is shown by the example

$$f(\zeta) = \left[\frac{1-r^2}{(1-\bar{z}\zeta)^2}\right]^{1/\delta}.$$

In the following theorem an estimate is given for the mean value of the modulus of a function, improving the estimate contained in the book of I. I. Privalov (([6]), p. 84).

Theorem 6. If $f(\zeta) \in H_\delta$, $0 < \delta < 1$, then

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|\, d\theta \leq \frac{1}{(1-r^2)^{1/\delta-1}}, \qquad 0 \leq r < 1. \tag{13}$$

Proof. Applying k times the Schwarz-Bunyakovsky inequality, we shall have

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|\, d\theta \leq \frac{1}{2\pi}(2\pi)^{1/2}\left(\int_0^{2\pi} |f(re^{i\theta})|^{2-\delta}\, d\theta\right)^{1/2} \leq \cdots$$

$$\cdots \leq \frac{1}{2\pi}(2\pi)^{\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^k}}\left(\int_0^{2\pi} |f(re^{i\theta})|^{2^k-(2^k-1)\delta}\, d\theta\right)^{2^{-k}}.$$

Letting $k \to \infty$, and using the relation

$$\lim_{\lambda\to\infty}\left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^\lambda\, d\theta\right)^{1/\lambda} = \max_{0\leq\theta\leq\pi} |f(re^{i\theta})|$$

and inequality (10), we obtain the estimate (13).

Consider the subclass $\widetilde{H}_1$ of functions $f(z)$ of the class H_1, representable in the form

$$f(z) = \sum_{k=1}^{\infty} c_k z^k = z\left(\sum_{k=0}^{\infty} b_k z^k\right)^2 = zF^2(z), \tag{14}$$

where the functions $F(z)$ are regular in the disk, $|c_0| \neq 0$ and fixed.

Theorem 7. The radii of univalence R_1 and of starlikeness ρ_1 for the class H_1 coincide and are equal to the root contained between 0 and 1 of the equation

$$\frac{9r^2 - 2r^4 + r^6}{1 + 6r^2 + r^4} = |c_1|. \tag{15}$$

The estimate is sharp, and equality is attained for the function

$$f(z) = z\left[b_0 - \frac{b_1}{3}\,\frac{3z - \rho_1 z^2}{(1 - \rho_1 z)^2}\right] \in \widetilde{H}_1, \tag{16}$$

where $b_0, b_1 > 0$, $b_0^2 = |c_1|$, and

$$b_1^2 = 9(1 - b_0^2)\,\frac{(1 - \rho_1^2)^3}{9 - 2\rho_1^2 + \rho_1^4}. \tag{17}$$

Proof. As is known, the disk $|z| < r$ is mapped by the function $w = f(z)$ onto a domain starlike with respect to the point $w = 0$, if for all z in this disk the inequality

$$\operatorname{Re}\left\{z\frac{f'(z)}{f(z)}\right\} \geq 0 \tag{18}$$

is satisfied.

Condition (18) is a consequence of the inequality

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq 1,$$

or, on the basis of (14), a consequence of the inequality

$$\sum_{k=1}^{\infty}(2k+1)|b_k|r^k \leq |b_0|. \tag{19}$$

Using the Cauchy-Schwarz inequality and noting that $\sum_{k=0}^{\infty}|b_k| \leq 1$, we obtain

$$\sum_{k=1}^{\infty}(2k+1)|b_k|r^k \leq \left[\frac{9r^2 - 2r^4 + r^6}{(1 - r^2)^3}\right]^{1/2}\sqrt{1 - |b_0|^2}.$$

Denote the root of equation (15) by ρ_1. The function $f(z) \in \widetilde{H}_1$ will also be univalent in the disk $|z| < \rho_1$.

The function (16), provided condition (17) is fulfilled, belongs to the class $\widetilde{H}_1$, and its derivative for $z = \rho_1$ vanishes. Hence, for this function $R_1 = \rho_1$, and ρ_1 is the radius of starlikeness.

The following theorem is proved analogously:

Theorem 8. The radius of convexity r_1 for the class of functions $f(z) \in H_2$, $f(0) = 0$, $|c_1| \neq 0$ fixed, is equal to the root contained between 0 and 1 of the equation

$$\frac{r^2(16 + r^2 - 11r^4 - 5r^6 + r^8)}{(1 - r^2)^5} = \frac{|c_1|^2}{1 - |c_1|^2}. \tag{20}$$

The estimate is sharp, and equality is attained for the function

$$f(z) = c_1 z - \frac{c_2}{4} \frac{4z^2 - 2r_1 z^3 + r_1^2 z^4}{(1 - r_1 z)^3}, \tag{21}$$

where

$$c_1 + \frac{c_2^2}{16} \frac{16 + r_1^2 + 11r_1^4 - 5r_1^6 + r_1^8}{(1 - r_1^2)^5} = 1, \qquad c_1 > 0, \quad c_2 > 0.$$

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Received
24 V 1962

References

1. L. V. Kresnyakova, *Izv. Vyssh. uchebn. zaved.*, Mathematics, No. 1, 98 (1961).
2. A. I. Macintyre, W. W. Rogosinsky, *Acta Math.*, 82, 275 (1950).
3. O. Szász, *Math. Zs.*, 8, 303 (1920).
4. G. M. Goluzin, *Tr. Mat. inst. im. V. A. Steklova AN SSSR*, 18 (1946).
5. Takenaka, *Tohoku Math. J.*, 27, 21 (1926).
6. I. I. Privalov, *Boundary Properties of Analytic Functions*, 1950.

Note: Figure translations are in progress. See original paper for figures.