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# MECHANICS

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1962

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## Abstract

## Full Text

MECHANICS

V. A. CHIKIN

# A GENERALIZATION OF A THEOREM OF HUYGENS

(Presented by Academician A. Yu. Ishlinskii on 20 X 1961)

§ 1. Let a rigid body move about a fixed point  $O$ . We shall consider the quantity of motion  $\mathbf{Q}$  and the kinetic moment  $\mathbf{K}_0$  of the body, according to their definition, as the principal vector and the principal moment of the system of vectors of the momenta  $m_i \mathbf{v}_i$  of its individual particles. The central axis of the screw of this system of vectors will be called the **kinetic axis of the body**. The point of intersection of this axis with the plane  $\alpha_\omega$ , passing through the center of gravity of the body  $G$  and the vector of the instantaneous angular velocity  $\bar{\omega}$ , will be called the **kinetic center of the body**  $A$ . If the radius vector of the point  $A$  is denoted by  $\mathbf{r}_A$ , then, according to the theory of reduction of a system of vectors <sup>(1)</sup>,

$$\mathbf{r}_A = \frac{\mathbf{Q} \times \mathbf{K}_0}{Q^2} = \frac{(\bar{\omega} \times \mathbf{r}_G) \times \mathbf{K}_0}{Mv_G^2}, \quad (1)$$

where  $M$  is the mass of the body;  $\mathbf{r}_G$  and  $\mathbf{v}_G$  are the radius vector and velocity of the center of mass  $G$ .

§ 2. Let us take a rectangular coordinate system  $Gx_1y_1z_1$  with origin at the center of gravity of the body  $G$  and with the axis  $Gy_1$  directed along the middle semiaxis of the central ellipsoid of inertia  $E_G$ . Then the equation of this ellipsoid can be written in the form

$$I_x^{(G)} x_1^2 + I_y^{(G)} y_1^2 + I_z^{(G)} z_1^2 - 2I_{zx}^{(G)} z_1 x_1 = 1, \quad (2)$$

where  $I_x^{(G)}$ ,  $I_y^{(G)}$ ,  $I_z^{(G)}$  are the axial moments, and  $I_{zx}^{(G)}$  is the centrifugal moment of inertia of the body with respect to the axes  $Gx_1y_1z_1$ .

The equation of the gyration ellipsoid  $E_G^{-1}$ , corresponding to the ellipsoid  $E_G$ , in the same axes will be

$$\frac{I_z^{(G)} x_1^2}{I_x^{(G)} I_z^{(G)} - I_{zx}^{(G)2}} + \frac{y_1^2}{I_y^{(G)}} + \frac{I_x^{(G)} z_1^2}{I_x^{(G)} I_z^{(G)} - I_{zx}^{(G)2}} + \frac{2I_{zx}^{(G)} z_1 x_1}{I_x^{(G)} I_z^{(G)} - I_{zx}^{(G)2}} = 1. \quad (3)$$

If the axis  $Gz_1$  is made to coincide with the perpendicular to one of the circular sections of the gyration ellipsoid, then the plane  $Gx_1y_1$  coincides with the plane of this section, and in equation (3) the coefficients of  $x_1^2$  and  $y_1^2$  become equal to each other.

Thus we obtain the relation

$$I_z^{(G)}(I_x^{(G)} - I_y^{(G)}) - I_{zx}^{(G)2} = 0, \quad (4)$$

which we shall call the **Hess structural condition**. When condition (4) is fulfilled, equations (2) and (3) are transformed into the form

$$I_y^{(G)}(x_1^2 + y_1^2) + \frac{1}{I_z^{(G)}}(I_z^{(G)}z_1 - I_{zx}^{(G)}x_1)^2 = 1, \quad (5)$$

$$\frac{1}{I_y^{(G)}}(x_1^2 + y_1^2) + \frac{I_x^{(G)}}{I_y^{(G)}I_z^{(G)}}z_1^2 + \frac{2I_{zx}^{(G)}}{I_y^{(G)}I_z^{(G)}}z_1x_1 = 1. \quad (6)$$

Now take at any point  $O$  of the perpendicular to the circular section  $E_G^{-1}$  chosen by us a coordinate system  $Oxyz$  with axes parallel to

axes  $Gx_1y_1z_1$ , setting the distance  $GO = z_G$ . In this case, on the basis of the theorems on moments of inertia with respect to parallel axes, the axial and centrifugal moments of inertia with respect to the system  $Oxyz$  will be

$$I_x = I_x^{(G)} + Mz_G^2, \quad I_y = I_y^{(G)} + Mz_G^2, \quad I_z = I_z^{(G)}, \quad (7)$$

$$I_{yz} = 0, \quad I_{zx} = I_{zx}^{(G)}, \quad I_{xy} = 0.$$

It follows from (7) that the axis  $Oy$  will be a principal axis of inertia, the structural condition (4) at the point  $O$  will remain exactly the same, and, consequently, the equation of the gyration ellipsoid  $E_0^{-1}$  at the point  $O$ , according to (6), will have the form

$$\frac{1}{I_y}(x^2 + y^2) + \frac{I_x}{I_y I_z'}z^2 + \frac{2I_{zx}}{I_y I_z'}zx = 1. \quad (8)$$

Comparing (8) with (6), we arrive at the conclusions:

- 1) The planes of the circular sections of the gyration ellipsoids constructed at the points of the perpendicular to the circular section of the central gyration ellipsoid are parallel to one another <sup>(2)</sup>.

- 2) Hess' s structural conditions for any heavy rigid body will be satisfied if it is fixed at any point lying on the perpendicular to one of the circular sections of the central gyration ellipsoid  $E_G^{-1}$  <sup>(3)</sup>.

§ 3. Let a heavy rigid body, fixed at a stationary point  $O$ , perform Hessian motion. Then the structural condition (4) must be satisfied, and the kinetic moment of the body  $\mathbf{K}_0$  will lie in the plane of the circular section of the gyration ellipsoid (8) <sup>(3)</sup>. These conditions, in the coordinate system  $Oxyz$  chosen by us, will be written in the form

$$x_G = y_G = 0, \quad z_G \neq 0; \quad K_z = -I_{zx}p + I_z r = 0. \quad (9)$$

Here  $x_G, y_G, z_G$  are the coordinates of the center of gravity of the body  $G$ ;  $p, q, r$  are the projections of the angular velocity  $\vec{\omega}$ .

Taking into account conditions (4) and (9), we find the position of the kinetic center according to (1):

$$\mathbf{r}_A = \frac{I_y}{Mz_G} \mathbf{k} = \frac{I_y}{Mz_G^2} \mathbf{r}_G = \frac{1}{\lambda} \mathbf{r}_G, \quad (10)$$

where  $\mathbf{k}$  is the unit vector of the axis  $Oz$ ;  $\lambda$  is a constant coefficient. Thus, in the motion of the body in the Hess case, the kinetic center will lie on the same perpendicular to the circular section  $E_0^{-1}$  as the center of gravity  $G$ .

It follows from (10) that the velocity of the kinetic center of the body is found as

$$\mathbf{v}_A = \vec{\omega} \times \mathbf{r}_A = \frac{1}{\lambda} \mathbf{v}_G. \quad (11)$$

Now multiplying (1) vectorially on the right by  $M\mathbf{v}_G$ , we obtain

$$\mathbf{K}_0 = \mathbf{r}_A \times M\mathbf{v}_G$$

or, taking (10) into account,

$$\mathbf{K}_0 = \mathbf{r}_A \times \lambda M\mathbf{v}_A = \lambda \mathbf{K}_0^A, \quad (12)$$

where  $\mathbf{K}_0^A$  denotes the moment of momentum of the kinetic center of the body relative to  $O$ , under the assumption that the mass of the body is concentrated in it.

Similarly, we express the moment of the weight of the body  $\mathbf{M}_0$  relative to the point  $O$ :

$$\mathbf{M}_0 = -Mg(\mathbf{r}_G \times \vec{\xi}^0) = -\lambda Mg(\mathbf{r}_A \times \vec{\xi}^0) = \lambda \mathbf{M}_0^A. \quad (13)$$

In this formula  $g$  denotes the acceleration of gravity;  $\vec{\xi}^0(\gamma_1, \gamma_2, \gamma_3)$  is the unit vector of the upward-directed vertical and  $\mathbf{M}_0^A$  is the moment of the weight of the ki-

of the kinetic center  $A$  relative to  $O$ , on the assumption that the mass of the body is concentrated in it.

§ 4. Using the preceding conclusions, we shall prove a theorem which is a generalization of Huygens' well-known theorem on the reciprocity of the point of suspension and the center of oscillation of a physical pendulum.

**Theorem.** *If the point of suspension  $O$  of a heavy rigid body satisfying Hesse' s conditions is moved to the kinetic center  $A$ , then the kinetic center moves to the point  $O$ , and the motion of the body remains unchanged under the same initial conditions.*

Substituting into the vector equation of motion of the body

$$\dot{\mathbf{K}}_0 + \vec{\omega} \times \mathbf{K}_0 = \mathbf{M}_0$$

the values of  $\mathbf{K}_0$  and  $\mathbf{M}_0$  according to (12) and (13), we obtain, after cancellation by  $\lambda$ ,

$$\dot{\mathbf{K}}_0^A + \vec{\omega} \times \mathbf{K}_0^A = \mathbf{M}_0^A. \quad (14)$$

We now project (14) onto the moving coordinate axes  $Oxyz$

$$\dot{p} - qr = \frac{g}{r_A} \gamma_2, \quad \dot{q} + pr = -\frac{g}{r_A} \gamma_1. \quad (15)$$

As the third equation we take Hesse' s fourth integral (9), which we write in the form

$$r = \frac{I_{zz}}{I_z} p. \quad (16)$$

In order that the system of equations (15) and (16) be closed, we add to them Poisson' s equations

$$\dot{\gamma}_1 = r\gamma_2 - q\gamma_3, \quad \dot{\gamma}_2 = p\gamma_3 - r\gamma_1, \quad \dot{\gamma}_3 = q\gamma_1 - p\gamma_2. \quad (17)$$

We now take the kinetic center  $A$  as the fixed point. In this case the distance from  $A$  to the center of gravity of the body will be, taking into account (10) and (7),

$$r_A - r_G = \frac{I_y}{Mz_G} - z_G = \frac{I_y - Mz_G^2}{Mz_G} = \frac{I_y^{(G)}}{Mz_G}. \quad (18)$$

If we denote by  $r'_A$  the distance from  $A$  to the new kinetic center and by  $I'_y$  the moment of inertia of the body about an axis passing through  $A$  parallel to the axis  $Oy$ , then, according to (10), (7), and (18), we obtain

$$r'_A = \frac{I'_y}{M(r_A - r_G)} = \frac{I_y^{(G)} + M(r_A - r_G)^2}{M(r_A - r_G)} = r_A, \quad (19)$$

i.e. the kinetic center coincides with the point  $O$ .

It is also evident from (7) that, when the point of suspension is displaced along the perpendicular to the circular section of the ellipsoid of gyration,  $I_{zx}$  and  $I_z$  will not change.

Thus, from the system of equations (15), (16), and (17) we conclude that, when the body is fixed at the point  $A$ , the general solution remains the same as when it is fixed at the point  $O$ . If, moreover, the same initial conditions remain, then the particular solution of this system will not change either.

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Received  
23 V 1960

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*Note: Figure translations are in progress. See original paper for figures.*

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