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**Abstract**

**Full Text**

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## ON THE BEHAVIOR OF AN ADIABATIC INVARIANT UNDER A SLOW PERIODIC CHANGE OF THE HAMILTONIAN FUNCTION

*(Presented by Academician A. N. Kolmogorov on 6 X 1961)*

§ 1. Let a dynamical system depend on a slowly varying parameter  $\lambda = \varepsilon t$ ; then the Hamiltonian function  $H(p, q; \lambda)$  contains the time  $t$  and is not conserved. A function  $J(p, q; \lambda)$  is called an *adiabatic invariant* of the system if, for small  $\varepsilon$ , the quantity

$$J(t) = J[p(t), q(t); \lambda(t)]$$

changes little over a time  $t \sim 1/\varepsilon$  (the changes in  $\lambda$  and  $H$  being finite).

Consider the phase plane  $p, q$  for a fixed value of the parameter  $\lambda$ . The level line of the energy  $H(p, q; \lambda) = H(p_0, q_0; \lambda)$  passing through the point  $p_0, q_0$  bounds a certain domain. Denote by  $2\pi I(p_0, q_0; \lambda)$  the area of this domain. It can be shown that  $I$  is an adiabatic invariant <sup>(1,2)</sup>.

From the adiabatic invariance of a quantity it does not follow, generally speaking, that it changes little for an unbounded time at fixed small  $\varepsilon$ . This is connected with the possibility of accumulation of small changes of the adiabatic invariant. Consider, for example, the linear oscillatory system

$$\ddot{x} = -\omega^2 x(1 + \alpha \cos \varepsilon t).$$

As is known, for certain  $\varepsilon$  (namely,  $\varepsilon \approx 2\omega/k$ ;  $k = 1, 2 \dots$ ) parametric resonance is possible, and  $I(t) \rightarrow \infty$  as  $t \rightarrow \infty$ . Here the rate of change of the system parameters  $\varepsilon$  can, evidently, be arbitrarily small.

It turns out, however, that *in a nonlinear system with a slowly periodically varying analytic Hamiltonian function  $H(p, q; \lambda)$  the adiabatic invariant is preserved forever*: for any  $\eta > 0$  there is an  $\varepsilon_0(\eta) > 0$  such that from  $|\varepsilon| < \varepsilon_0$  it follows that

$$|I(t) - I(0)| < \eta$$

for all  $-\infty < t < \infty$ .

The linear system occupies an exceptional position because the frequency of its oscillations does not depend on the amplitude. In a nonlinear system, however, as the amplitude increases the frequency changes, and the oscillations do not have time to grow before the resonance condition  $\varepsilon \approx 2\omega/k$  is violated.

The proof of the eternal adiabatic invariance of the action is outlined in the following paragraphs. By an analogous method one can show that the adiabatic invariant  $I_y$  is eternally preserved for an autonomous oscillatory system with two degrees of freedom and analytic Hamiltonian function

$$H = \frac{\dot{x}^2 + \dot{y}^2 + U(\varepsilon x, y)}{2}.$$

It is only necessary that, in the first approximation, the ratio of the frequencies  $\omega_x/\omega_y$  depend, for fixed total energy  $H = h$ , on the amplitude of the  $y$ -oscillations.

In particular, a field with potential

$$U = \omega^2 y^2 \quad (\omega = 1 + \lambda^2, \lambda = \varepsilon x)$$

is, for  $\varepsilon \ll 1$ , a trap capable of forever holding a particle with initial conditions  $x_0, y_0, \dot{x}_0, \dot{y}_0$  of order 1. This follows from the eternal adiabatic invariance of the quantity

$$I_y = \frac{\dot{y}^2 + U}{2\omega}.$$

\* *Note added in proof.* By the same method one can prove the eternal adiabatic invariance of the magnetic moment in an axially symmetric magnetic trap <sup>(6)</sup>.

§ 2. Let the analytic Hamiltonian function  $H$  of an oscillatory system with one degree of freedom depend on the slow time  $\lambda = \varepsilon t$  periodically with period  $2\pi$ . For a fixed ( $\dot{\lambda} = 0$ ) value of the parameter  $\lambda$ , in the action-angle variables  $I, \varphi$  the function  $H$  has the form  $H_0(I; \lambda) = H_0(I; \lambda + 2\pi)$ ,  $\dot{I} = 0$ , and the torus  $I = \text{const}$  in the space  $p, q, \lambda$  (where  $p, q, \lambda$  and  $p, q, \lambda + 2\pi$  are identified) is invariant. The phase point moves along a meridian of the torus ( $\lambda = \text{const}$ ) with frequency

$$\dot{\varphi} = \frac{\partial H_0}{\partial I} = \omega(I, \lambda).$$

Denote by  $\bar{\omega}(I)$  the mean frequency  $\frac{1}{2\pi} \oint \omega(I, \lambda) d\lambda$ , and denote the torus itself by  $T_{\bar{\omega}}$ .

If  $\varepsilon \neq 0$  is small, then in the first approximation one may assume that the motion takes place on the torus  $T_{\bar{\omega}}$ , with the longitude  $\lambda$  changing slowly ( $\dot{\lambda} = \varepsilon$ ), while the latitude  $\varphi$  changes rapidly with slowly varying frequency  $\dot{\varphi} = \omega(I, \lambda)$ . It turns out that, for sufficiently small  $\varepsilon$  and for a frequency  $\bar{\omega}$  "sufficiently incommensurable" with  $\varepsilon$ , there actually exists an invariant torus  $T_{\bar{\omega}}(\varepsilon)$  close to  $T_{\bar{\omega}}$ . This torus is filled with conditionally periodic trajectories with frequencies  $\bar{\omega}$  and  $\varepsilon$ .

For fixed  $\varepsilon$ , the invariant two-dimensional tori  $T_{\bar{\omega}}(\varepsilon)$  divide the three-dimensional space into thin toroidal layers. Each trajectory beginning in such

a layer is entirely contained in it. The thickness of the layers tends to zero together with  $\varepsilon$ ; therefore, after the tori  $T_{\bar{\omega}}(\varepsilon)$  have been found, the proof of the eternal adiabatic invariance of  $I$  is not difficult.

These tori are sought by means of Newton's method <sup>(3)</sup>. In the corresponding Fourier series there appear "small denominators"  $m\omega + n\varepsilon$ . Some of them are small because of the approximate commensurability of the frequencies  $\bar{\omega}$  and  $\varepsilon$ , while others are small because the frequency  $\varepsilon$  is small (degeneracy). The difficulty connected with the degeneracy is overcome on the basis of the same considerations as in note <sup>(4)</sup>, which is the non-Hamiltonian analogue of the present work.

**§ 3. Preliminary canonical transformation.** In the action-angle variables  $I, \varphi$  the Hamiltonian function has, as is known <sup>(1,2)</sup>, the form

$$H(I, \varphi, \lambda) = H_0(I, \lambda) + \varepsilon H_1(I, \varphi, \lambda) \quad (\lambda = \varepsilon t). \quad (1)$$

We shall assume that the functions  $H_0$  and  $H_1$ , having periods  $2\pi$  in  $\varphi$  and  $\lambda$ , are analytic in a complex neighborhood of the toroidal layer  $I_1 \leq I \leq I_2$ .

**Theorem 1.** *Let the frequency  $\omega(I, \lambda) = \partial H_0 / \partial I$  not vanish in the layer under consideration. Then there exist positive numbers  $\varepsilon_0, r_0, \rho_0$  and analytic functions  $P, Q, T$  of the variables  $I, \varphi, \lambda$ , independent of  $\varepsilon$ , such that:*

1. *The functions  $P, Q - \lambda$ , and  $T - \varphi$  have periods  $2\pi$  in  $\varphi$  and  $\lambda$ .*
2. *The canonical equations with the Hamiltonian function (1) are equivalent to the canonical equations with the Hamiltonian function*

$$k(P, Q, T) = \varepsilon k_0(P) + \varepsilon^2 k_1(P, Q, T) + \dots, \quad (2)$$

*analytic for  $|\varepsilon| \leq \varepsilon_0, |\operatorname{Im} Q, T| \leq \rho_0, |P - P_0| \leq r_0$ , and having periods  $2\pi$  in  $Q$  and  $T$ .*

3. *The principal part of the Hamiltonian function (2) is  $\varepsilon k_0(P)$ , where the function  $k_0(P)$  is the inverse of*

$$\bar{H}_0(I) = \frac{1}{2\pi} \oint H_0(I, \lambda) d\lambda,$$

*so that  $\bar{H}_0(k_0(P)) \equiv P$ .*

We first introduce the new time  $T = \varphi$ . As is known <sup>(5)</sup>, the integral curves of a Hamiltonian system in the space  $I, \varphi, \lambda$  are invariantly connected with the differential form

$$I d\varphi - H(I, \varphi, \varepsilon t) dt = -\frac{1}{\varepsilon} (H d\lambda - \varepsilon I d\varphi). \quad (3)$$

Multiplication of the form by a constant does not change this connection. We shall regard

in (3) the independent variables are not  $I, \omega, \lambda$ , but  $H, \lambda, \dots$ . Solving (1) with respect to  $I$ , we obtain

$$I(H, \lambda, \omega) = I_0(H, \lambda) + \varepsilon I_1(H, \lambda, \omega) + \dots \quad (4)$$

Introduce the notation\*  $p = H, q = \lambda, T = \omega, K = \varepsilon I$ , so that

$$K(p, q, T) = \varepsilon I_0(p, q) + \varepsilon^2 I_1(p, q, T) + \dots \quad (5)$$

Then

$$H d\lambda - \varepsilon I d\omega = p dq - K(p, q, T) dT,$$

and therefore (5) the systems with Hamiltonian functions (1) and (5) are equivalent.

Let us note that the frequency  $\omega(I, \lambda)$  in § 2 varied with time. By means of a canonical transformation  $p, q \rightarrow P, Q$  we change the coordinate  $q = \lambda$  (which has the meaning of time) so that the frequency with respect to the changed time  $Q$  becomes constant,  $\bar{\omega}(I)$ . For this purpose we introduce in the system with Hamiltonian function  $I_0(p, q)$  the variables action  $P$ —angle  $Q$ .

If  $S(q, P)$  is the generating function, then the transformation is determined by the equations

$$p = \frac{\partial S}{\partial q}; \quad Q = \frac{\partial S}{\partial P}.$$

Choose  $S$  so as to satisfy the Hamilton-Jacobi equation

$$I_0\left(\frac{\partial S}{\partial q}, q\right) = k_0(P)$$

with the function  $k_0(P)$  still unknown. According to (4) we find  $\partial S / \partial q = H_0(k_0(P), q)$ , or

$$S = \int^q H_0(I, \lambda) d\lambda, \quad \text{where } I = k_0(P). \quad (6)$$

The periodicity condition  $Q(p, q + 2\pi) = Q(p, q) + 2\pi$  now gives

$$\oint \frac{\partial H_0}{\partial I} \frac{dk_0}{dP} d\lambda = 2\pi; \quad \frac{dk_0}{dP} \frac{d\bar{H}_0}{dI} = 1. \quad (7)$$

Equality (7) will be satisfied if for  $k_0(P)$  one takes the function inverse to  $\bar{H}_0(I)$ . Then the generating function (6) introduces the variables  $P, Q$  satisfying theorem 1. In this case  $k(P, Q, T) = K(p, q, T)$ .

§ 4. The construction of invariant tori of the system with Hamiltonian function (2) is carried out by successive approximations of Newton type <sup>(3, 4)</sup>. Let  $\varepsilon \neq 0$ . We shall say that the number  $\bar{\omega}$  is sufficiently incommensurable with  $\varepsilon$ , and shall write  $\bar{\omega} \in \Omega(\varepsilon)$ , if

$$|m\bar{\omega} + n\varepsilon| > |\varepsilon|(|m| + |n|)^{-2}$$

for all integers  $m, n$ ,  $|m| + |n| > 0$ . Denote by  $\bar{\Omega}(\varepsilon)$  the complement to  $\Omega(\varepsilon)$  on the  $\omega$ -axis. It is easily proved:

**Lemma.** *The measure of  $\bar{\Omega}(\varepsilon)$  does not exceed  $10\varepsilon$ .*

To each value  $P = P_1$  there corresponds  $I_1 = k_0(P_1)$  and a definite frequency  $\bar{\omega}_1 = \omega(I_1)$ . Suppose that  $d\bar{\omega}/dI \neq 0$  (which is equivalent to the conditions  $d^2\bar{H}_0/dI^2 \neq 0$ ;  $d^2k_0/dP^2 \neq 0$ ). From the lemma just stated it is not difficult to infer that the measure of the set of those  $P_1$  for which  $\bar{\omega}_1 \in \bar{\Omega}(\varepsilon)$  tends to zero together with  $\varepsilon$ .

**Theorem 2.** *Let the Hamiltonian function (2) be analytic for  $|\varepsilon| < \varepsilon_0$  in the following neighborhood of the torus  $P = P_1$ :*

$$|P - P_1| \leq r_1, \quad |\operatorname{Im} Q, T| \leq \rho_0.$$

*Suppose that in this neighborhood the inequalities*

$$|k_0| \leq M, \quad |k_1 + \varepsilon k_2 + \dots| \leq M, \quad \left| \frac{d^2 k_0}{dP^2} \right| \geq \theta > 0.$$

\* Do not confuse with  $p, q$  from §§ 1, 2!

Then there exists an  $\varepsilon_1(r_1, \rho_0, M, \theta) > 0$  such that, if for some  $\varepsilon < \varepsilon_0, \varepsilon_1$  the frequency  $\omega_1$  is sufficiently incommensurable with  $\varepsilon$ , then there exist analytic functions  $F_\varepsilon(Q, T), G_\varepsilon(Q, T)$  such that the torus  $P = F_\varepsilon(Q, T)$  is invariant and on it  $dG_\varepsilon/dT = \varepsilon/\omega_1$ . The functions  $F_\varepsilon(Q, T)$  and  $G_\varepsilon(Q, T) - Q$  have periods  $2\pi$  in  $Q$  and  $T$  and, as  $\varepsilon \rightarrow 0$ , tend to  $P_1$  and to zero respectively.

The proof does not fit within the scope of the present note (see <sup>(3, 4)</sup>). As explained in § 2, Theorem 2 implies

**Theorem 3.** *Let an oscillatory system have, in action-angle variables, an analytic Hamiltonian function (1), and let throughout the toroidal layer  $I_1 \leq I \leq I_2$  one have  $\partial H_0/\partial I \neq 0$ ,  $d^2 H_0/dI^2 \neq 0$ . Then for any  $\eta > 0$  there is an  $\varepsilon_2 > 0$  such that, if  $|\varepsilon| < \varepsilon_2$  and  $I_1 + \eta \leq I(0) \leq I_2 - \eta$ , then for all  $-\infty < t < +\infty$  one has  $|I(t) - I(0)| < \eta$ .*

§ 5. Theorem 3 is also valid in the case when the Hamiltonian function changes conditionally periodically, namely, when in the variables of § 1  $H(p, q, \lambda_1, \dots, \lambda_n)$  depends on several angular parameters  $\lambda$ , each varying with its own frequency  $\dot{\lambda}_i = \varepsilon \mu_i$ . Suppose that the  $\mu_i$  are strongly incommensurable:

$$\left| \sum \mu_i n_i \right| > C \left( \sum |n_i| \right)^{-\nu}, \quad \text{if } \sum |n_i| > 0, \quad (8)$$

for some  $C, \nu > 0$ .

Passing to "time"  $T = \omega$ , we obtain the Hamiltonian function (5) in the form  $K(h, q, T)$ , where  $h = \sum \mu_i p_i$ . By virtue of condition (8), the transformation  $p, q \rightarrow P, Q$  is possible. The  $P_i$  will enter the Hamiltonian function (2) only in the form of the combination  $H = \sum \mu_i P_i$ . Analogously to Theorem 2, one can find invariant manifolds  $H = F_\varepsilon(Q, T)$ ; to them there correspond invariant  $(n+1)$ -dimensional tori in the original  $(n+2)$ -dimensional space  $p, q, \lambda_1, \dots, \lambda_n$ .

§ 6. The case of several degrees of freedom presents considerable difficulties. Theorem 2 admits the needed generalization, but, generally speaking, it is not possible to reduce the Hamiltonian function to the form (2). The point is that the ratio of the frequencies of the fast motions depends on the phase of the slow motion. The system of equations on the three-dimensional torus

$$\dot{x} = \mu_1(z) + \varepsilon f(x, y, z); \quad \dot{y} = \mu_2(z) + \varepsilon g(x, y, z); \quad \dot{z} = \varepsilon \quad (9)$$

( $x, y, z$  are the angular coordinates of a point of the torus) gives a simple example of this phenomenon; the trajectories (9) cannot be straightened by a change of variables that is small together with  $\varepsilon$ .

Therefore, in the case of a general system with  $n$  separating variables and slowly periodically varying coefficients, there are hardly any tori filled with conditionally periodic trajectories. Even if such  $(n+1)$ -dimensional tori were found, they would not divide the  $(2n+1)$ -dimensional space  $p_i, q_i, \lambda$  and would not make it possible to prove the eternal adiabatic invariance of the action variables.

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*Note: Figure translations are in progress. See original paper for figures.*

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