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Abstract

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A SYSTEM OF QUASILINEAR EQUATIONS OF THE THEORY OF SURFACES

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In this note the fundamental equations of the theory of surfaces are investigated. The equations of differential geometry that determine, in Euclidean space, a surface with a prescribed metric can be reduced to a system of two quasilinear differential equations in two independent variables. The introduction of the Riemann invariants as new variables leads to a simple form of these equations, having a clear geometric meaning. From this there follow some results for surfaces of negative Gaussian curvature.

Let the metric of the surface be given in the form

$$ds^2 = dx^2 + B^2(x, y)\, dy^2. \tag{1}$$

Here the lines $y = \text{const}$ are geodesics, $x = \text{const}$ is the family of trajectories orthogonal to them, and the coordinate system (x, y) itself is called semigeodesic*. In this case the Gaussian curvature $K(x, y)$ of the surface is determined by the formula

$$B''_{xx}(x, y) + K(x, y)B(x, y) = 0, \tag{2}$$

and the question of realizing a surface with such a metric in Euclidean space is determined by the existence of the coefficients of the second quadratic form L, M, N.

These coefficients must satisfy the Peterson-Codazzi equations (^{1})

$$M_x - L_y = -\frac{B_x}{B} M, \qquad N_x - M_y = \frac{B_x}{B}\,(N + B^2 L) - \frac{B_y}{B} M \tag{3}$$

and the Gauss formula

$$LN - M^2 = B^2 K. \tag{4}$$

Formula (4) makes it possible to eliminate from the system (3) one of the unknowns L, M, N, after which the system (3) becomes a system of two quasilinear equations in two independent variables. Substituting in (3) $L = (M^2 + B^2 K)/N$, and then investigating the system of equations (3), it is easy to establish that it is elliptic for $K > 0$, parabolic for $K = 0$, and hyperbolic for $K < 0$.

In the case $K < 0$ the characteristics of this system of quasilinear equations are determined by the equation

$$L dx^2 + 2M dx\, dy + N dy^2 = 0, \tag{5}$$

i.e., they are the asymptotic lines of the surface.

Reducing this system to characteristic form, we take as new unknowns the Riemann invariants (2), given by the formulas

$$r = B\,\frac{-M - B\sqrt{-K}}{N}, \qquad s = B\,\frac{-M + B\sqrt{-K}}{N}. \tag{6}$$

After this, system (3) is written in the simple form

$$\frac{\partial r}{\partial x} + \frac{s}{B}\frac{\partial r}{\partial y} = -s(1+r^2)\frac{B_x}{B} + \frac{r-s}{2}\left[\frac{\partial Q}{\partial x} + \frac{r}{B}\frac{\partial Q}{\partial y}\right],$$

$$\frac{\partial s}{\partial x} + \frac{r}{B}\frac{\partial s}{\partial y} = -r(1+s)^2\frac{B_x}{B} + \frac{s-r}{2}\left[\frac{\partial Q}{\partial x} + \frac{s}{B}\frac{\partial Q}{\partial y}\right], \tag{7}$$

where $Q = \ln\sqrt{-K(x,y)}$.

The system of quasilinear equations (7) is weakly nonlinear (2), while r and s are the tangents of the angles formed on the surface by the direction of the asymptotic lines (characteristics) with the geodesic lines $y = \text{const}$. Introducing these angles themselves as the unknowns, one may also rewrite system (7) in the form

$$\frac{\partial \varphi_1}{\partial s_2} = -\sin\varphi_2\frac{B_x}{B} + \frac{1}{2}\sin\omega\,\frac{\partial Q}{\partial s_1};$$

$$\frac{\partial \varphi_2}{\partial s_1} = -\sin\varphi_1\frac{B_x}{B} - \frac{1}{2}\sin\omega\,\frac{\partial Q}{\partial s_2}; \qquad \omega = \varphi_1 - \varphi_2, \tag{8}$$

where

$$\frac{\partial}{\partial s_2} = \cos\varphi_2\frac{\partial}{\partial x} + \frac{\sin\varphi_2}{B}\frac{\partial}{\partial y}; \qquad \frac{\partial}{\partial s_1} = \cos\varphi_1\frac{\partial}{\partial x} + \frac{\sin\varphi_1}{B}\frac{\partial}{\partial y} \tag{9}$$

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* The convenience of using the semigeodesic coordinate system was pointed out to us by N. V. Efimov.

are the differentiation operators in the direction of the asymptotic lines.

Such a form of the basic equations of the theory of surfaces, in characteristic form with the use of Riemann invariants, may prove useful in studying the question of immersing a surface of prescribed negative Gaussian curvature in Euclidean space.

As an example, we give a simple proof of Hilbert' s theorem (3) on the impossibility, in Euclidean space, of a complete regular surface of constant negative curvature, based entirely on equations (7), (8).

Suppose that a complete regular surface with Gaussian curvature $K = \text{const} < 0$ is immersed in Euclidean space. In this case $\partial Q/\partial s_1 = \partial Q/\partial s_2 = 0$, and from equations (9), (8) the commutativity of the operators $\partial/\partial s_1$ and $\partial/\partial s_2$ follows. Hence it follows that the length of a segment of a characteristic of one family, contained between two characteristics of the other family, is constant. We have obtained the well-known property of surfaces with constant negative curvature: the asymptotic lines form a Chebyshev net.

Let us establish the following property of the asymptotic lines ($K = \text{const} < 0$): the area of the surface enclosed between a segment of a geodesic line drawn through any two points A and B of an asymptotic line, and this asymptotic line, does not exceed $2\pi/|K|$.

Indeed, suppose that through A and B there passes an asymptotic line B, $dy/dx = s$, and the geodesic $y = 0$. Integrating the first equation (8), we obtain:

$$\varphi_1(B) - \varphi_1(A) = -\int_A^B \frac{B_x}{B} \sin\varphi_2 \, ds_2 = -\int_A^B B_x(x, y)\, dy =$$

$$= -\oint_{C_{AB}} B_x(x, y)\, dy = K \iint_{G_{C_{AB}}} B \, dx\, dy = K S_{C_{AB}}.$$

Here C_{AB} is the contour bounded by the asymptotic AB and the segment of the geodesic joining the points A, B (possibly with self-intersection); $G_{C_{AB}}$ is the domain bounded by C_{AB}; $S_{C_{AB}}$ is the area of the surface bounded–

bounded by the contour C_{AB}; here the sign of the area is taken to be different for parts of the surface lying on different sides of the geodesic.

For a regular surface of negative curvature, the directions of asymptotic lines of different families never coincide; therefore it is easy to establish that $|\varphi_1(B) - \varphi_1(A)| < 2\pi$. From this follows the property formulated above.

Combining this property with the fact that asymptotic lines form a Chebyshev net, one can easily establish that in Euclidean space there exists no complete regular surface with constant negative Gaussian curvature.

Let us establish an analogous property of asymptotic lines also in the case of variable Gaussian curvature $K = K(x) \leq -K_0 < 0$, which, however, is formulated in a more complicated way because the coordinate system is not invariant.

Suppose an asymptotic line is not orthogonal to the geodesics $y = \text{const}$. Then from the system of equations (8) it follows that, under certain restrictions on the quantities K_x, K_{xx}, an asymptotic line cannot simultaneously intersect two rays of geodesics $y = 0$ $(x \geq x_0)$ and $y = h$ $(x \geq x_0)$ when $B'(x_0) > 4\pi/h$.

As a very simple consequence of formulas (8), we note the inequality, valid for arbitrary curvature $K(x, y) \leq -K_0 < 0$,

$$S_{C_{AB}} < \frac{4\pi + HL_{AB}}{2K_0},$$

where $S_{C_{AB}}$ retains its former meaning; L_{AB} is the length of the segment of the asymptotic line between the points A and B, and $|\operatorname{grad} Q| \leq H$.

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[3] D. Hilbert, *Foundations of Geometry*, Moscow-Leningrad, 1948.

Note: Figure translations are in progress. See original paper for figures.

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