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Abstract

Full Text

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ON THE INTEGRAL REPRESENTATION OF CONTINUOUS HERMITIAN-INDEFINITE KERNELS

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Let χ be some nonnegative integer ($\chi < \infty$). A continuous Hermitian kernel $K(x, y)$, $x, y \in (a, b)$ $(-\infty \le a, b \le +\infty)$, will be called Hermitian-indefinite with χ negative squares (h.i.) if, for any $x_1, \dots, x_n \in (a, b)$, the form

$$\sum_{j,k=1}^{n} K(x_j, x_k)\xi_k\overline{\xi_j} \tag{1}$$

has no more than χ negative squares, and at least one such form has exactly χ negative squares.

In the case of kernels of the form $K(x, y) = k(y-x)$ $(-\infty < x < \infty)$, M. G. Krein ([1]) obtained an integral representation of h.i. functions $k(t)$ of Bochner-theorem type. Below this result is generalized to the case of kernels that are expanded in eigenfunctions of a differential operator. In the case of a difference operator, a similar representation was obtained by the author in ([2])* by developing the method of M. G. Krein and I. S. Iokhvidov ([3]) for h.i. sequences c_j. The method of proof in this note is a continual variant of the reasoning in ([2]); it also uses the theory of representation of positive definite kernels developed by M. G. Krein ([4]) and Yu. M. Berezanskii ([5]). We note that in this note, in particular, the problem of extending h.i. functions from a finite interval to the whole axis is solved without additional restrictions.

1. Let $K(x, y)$ be h.i. We associate with it a space of type Π_χ ([6]). To this end, consider the manifold $C_0^\infty(a, b)$ of finite infinitely differentiable functions on (a, b) with scalar product

$$\langle \varphi, \psi \rangle = \int_a^b \int_a^b K(x, y)\varphi(y)\overline{\psi(x)}\, dx\, dy. \tag{2}$$

It follows from the definition of an h.i. function that there exist χ linearly independent functions of the form

$$\xi_m(x) = \sum_j \xi_j^{(m)} \delta(x - x_j) \quad (m = 1, \dots, \chi),$$

for which

$$\int_a^b \int_a^b K(x,y)\xi_m(y)\overline{\xi_n(x)}\,dx\,dy = \sum_{j,k} K(x_j,x_k)\xi_k^{(m)}\overline{\xi_j^{(n)}} = \begin{cases} -1 & \text{when } m = n, \\ 0 & \text{when } m \neq n. \end{cases} \tag{3}$$

Since every generalized function can be approximated by finite infinitely differentiable functions, in $C_0^\infty(a,b)$ there will be found—

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* The requirement imposed in that article that, among $\det \|F_{jk}\|_{j,k=1}^n$, for large n, there occur values different from zero can be dropped. The results of that work are also valid for the case of a differential expression with variable coefficients, owing to the possibility of extending a Hermitian operator to a self-adjoint one with passage to a broader space of type Π_χ with the same χ. This fact was communicated to the author by M. G. Krein.

there are $\varkappa$ linearly independent negative functions $e_1, \dots, e_\varkappa$: $\langle e_m, e_m \rangle < 0$. Applying to them the usual orthogonalization process, we obtain a basis of a negative $\varkappa$-dimensional subspace in $C_0^\infty(a,b)$. It also follows from the definition of an h.i. function that in $C_0^\infty(a,b)$ there do not exist negative subspaces of dimension greater than $\varkappa$. Identifying in $C_0^\infty(a,b)$ functions φ and ψ such that $\varphi - \psi$ is an element of an isotropic subspace, and using Theorem 1.4 ([6]), we obtain that $C_0^\infty(a,b)$ can be completed to a space of type $\Pi_\varkappa$.

2. Consider on (a,b) an ordinary differential expression

$$\mathcal{L}[u] = \sum_{0 \le k \le r} a_k(x) \frac{d^k u}{dx^k} \tag{4}$$

with complex-valued coefficients, each of which, for simplicity of exposition, we shall assume to be infinitely differentiable. Suppose that there exists an h.i. kernel $K(x,y)$ for which

$$\mathcal{L}_x[K(x,y)] = \overline{\mathcal{L}}_y[K(x,y)] \tag{5}$$

(the bar denotes passage to the complex-conjugate coefficients; the equality is understood in the sense of Schwartz distributions).

Theorem 1. *To every h.i. kernel for which (5) holds there corresponds at least one polynomial $P(\lambda)$ of degree $\varkappa$ such that the kernel, generalized in the sense of Schwartz,*

$$\Phi(x,y) = \overline{P}(\overline{\mathcal{L}})_y \, P(\overline{\mathcal{L}})_x [K(x,y)]$$

is positive definite.

We outline the proof. The operator L_0, defined on $C_0^\infty(a,b)$ by means of the differential expression $\mathcal{L}^+$ (+ denotes passage to the expression formally adjoint to $\mathcal{L}$), is Hermitian in $\Pi_\varkappa$, constructed from $K(x,y)$. It is not difficult to see that this definition of L_0 is correct. The operator L_0 can be extended to a self-adjoint operator L, generally speaking, with range in a wider space of type $\Pi_\varkappa$ with the same $\varkappa$.* By a theorem of L. S. Pontryagin ([8]), for L there exist an invariant nonnegative $\varkappa$-dimensional subspace $\mathcal{T}$, in which all eigenvalues of the operator L have nonnegative imaginary part, and $\mathcal{T}'$, an invariant nonnegative subspace of dimension $\varkappa$, in which all eigenvalues of L have nonpositive imaginary part. Let $P(\lambda)$ and $\overline{P}(\lambda)$ be the characteristic polynomials of L in $\mathcal{T}$ and $\mathcal{T}'$, respectively. The operator $P(L')$, where L' is the restriction of L to $\mathcal{T}+\mathcal{T}'+C_0^\infty(a,b)$, annihilates $\mathcal{T}$, and $\overline{P}(L')$ annihilates $\mathcal{T}'$. Therefore $M=\{P(\mathcal{L}^+)\varphi\}$, $\varphi\in C_0^\infty(a,b)$, is orthogonal to $\mathcal{T}'$. By Theorem 1.2 ([6]), M, as orthogonal to a $\varkappa$-dimensional nonnegative subspace in $\Pi_\varkappa$, is itself nonnegative, i.e.

$$\langle P(\mathcal{L}^+)\varphi,\ (\mathcal{L}^+)\varphi\rangle \geq 0 \quad (\varphi\in C_0^\infty(a,b)),$$

and this means precisely that

$$\Phi(x,y)=\overline{P}(\overline{\mathcal{L}})_y\,P(\overline{\mathcal{L}})_x[K(x,y)]$$

is positive definite.

3. Since

$$\Phi(x,y)=\overline{P}(\overline{\mathcal{L}})_y\,P(\overline{\mathcal{L}})_x[K(x,y)]$$

is a positive definite generalized kernel, using the works ([5],[9]) (or the method of directing functionals ([11])), we obtain the representation, in the weak sense,

$$\overline{P}(\overline{\mathcal{L}})_y\,P(\overline{\mathcal{L}})_x[K(x,y)]=\int_{-\infty}^{\infty}\sum_{j,k=0}^{r-1}\chi_j(x,\lambda)\,\overline{\chi_k(y,\lambda)}\,d\rho_{j,k}(\lambda), \tag{6}$$

where $\chi_j(x,\lambda)$ $(j=0,1,\dots,r-1)$ is a fundamental system of solutions of the equation $\mathcal{L}[v]=\lambda v$, satisfying initial data of the form

$$\left.\frac{d^k}{dx^k}\chi_j(x,\lambda)\right|_{x=a}=\delta_{j,k}\quad (k=0,1,\dots,r-1)$$

(a is an arbitrary point),

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* The possibility of such an extension was communicated to the author by M. G. Krein and I. S. Iokhvidov.

$\|d\rho_{j,k}(\lambda)\|_{j,k=0}^{r-1}$ is a matrix distribution function. It remains to solve the differential equation (6) with respect to $K(x,y)$.

Theorem 2. Let $\mathcal{L}$ be an ordinary differential expression of the form (4), and let $K(x,y)$ be an H.-i. kernel for which (5) holds. Then $K(x,y)$ admits the integral representation

$$K(x,y)=T_\rho(x,y)+\int_{-\infty}^{\infty}\frac{\sum\limits_{j,k=0}^{r-1}\chi_j(x,\lambda)\overline{\chi_k(y,\lambda)}-S_\rho^{(j,k)}(x,y,\lambda)}{Q_0^2(\lambda)}\,d\sigma_{j,k}(\lambda); \quad (7)$$

here $\|d\sigma_{j,k}(\lambda)\|_{j,k=0}^{r-1}$ is a matrix distribution function,

$$Q_0(\lambda)=\prod_{n=1}^{s}(\lambda-a_n)^{\rho_n},$$

where $a_1,\dots,a_n$ are all the real distinct zeros of the polynomial $P(\lambda)$, and ρ_n are their multiplicities; $S_\rho^{(j,k)}(x,y,\lambda)$ are any corrections regularizing the integrals; $T_\rho(x,y)=\overline{T_\rho(y,x)}$ is a suitably chosen Hermitian solution of the equation

$$\overline{P(\mathcal{L})_y}\,P(\mathcal{L})_x\,u(x,y)=0.$$

As in (1), the function $S_\rho^{(j,k)}(x,y,\lambda)$ $(\rho>0)$ will be called a regularizing correction in the integrals (7) if $S_\rho^{(j,k)}(x,y,\lambda)=0$ for $|\lambda|>\rho$, while for $|\lambda|\le\rho$ the function $S_\rho^{(j,k)}(x,y,\lambda)$ is equal to the product $Q_0^2(\lambda)$ by the sum of the principal parts of the function $\chi_j(x,\lambda)\overline{\chi_k(y,\lambda)}/Q_0^2(\lambda)$ with respect to all its poles.

4. Let us consider two examples.

Example 1. Let $k(x)$ $(-\infty<x<+\infty)$ be a continuous function such that the kernel $K(x,y)=k(x+y)$ is H.-i. For simplicity we shall take $\chi=1$. The expression $\mathcal{L}[u]=du/dx$ satisfies (5). It is not difficult to write a representation for $K(x,y)=k(x+y)$. Then, replacing $x+y$ by t, for $k(t)$ we obtain: either

$$k(t)=(At+B)e^{a_1t}+\int_{-\infty}^{\infty}\frac{e^{\lambda t}-e^{a_1t}[t(a_1+\lambda)+1]}{(\lambda-a_1)^2}\,d\sigma_{0,0}(\lambda),$$

or

$$k(t)=Ce^{\alpha_1t}+De^{\bar{\lambda}_1t}+\int_{-\infty}^{\infty}e^{\lambda t}\,d\sigma_{0,0}(\lambda),$$

where A,B,C,D are certain constants, and λ_1 is a certain non-real number.

Example 2. Let $k(x)$, $-\infty<x<+\infty$, be a continuous function such that the kernel $K(x,y)=k(y-x)$ is H.-i. Then $\mathcal{L}[u]=i\,du/dx$ satisfies (5), and for $k(x)$ we obtain the representation, given by M. G. Krein in ([1]):

$$k(x) = h_\rho(x) + \int_{-\infty}^{\infty} \frac{e^{i\lambda x} - S_\rho(x,\lambda)}{Q_0^2(\lambda)}\, d\sigma(\lambda), \tag{8}$$

where $h_\rho(x)$ is a suitably chosen Hermitian solution of the equation

$$P\left(-i\frac{d}{dx}\right)\overline{P}\left(-i\frac{d}{dx}\right)h = 0.$$

In the case of an H.-i. kernel on a one-dimensional interval $(-a, a)$, as is clear from the preceding, we obtain the same representation (8). The existence of such a representation shows that every H.-i. function on a finite interval can be extended to an H.-i. function on the whole axis.

5. The result stated above can be generalized to the case of kernels $K(x, y)$ defined for $x, y \in G$, where G is a domain of n-dimensional space—

properties; in this case $\mathcal{L}$ is a partial differential expression. Analogously to the preceding, one can prove that the kernel $\Phi(x,y) = \overline{P(\mathcal{L})_y}\, P(\mathcal{L})_x[K(x,y)]$ is positive definite and therefore admits an integral representation. In obtaining a representation of the kernel $K(x, y)$, difficulties arise on which we shall not dwell here.

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REFERENCES

1. M. G. Krein, DAN, **125**, No. 1 (1959).
2. V. I. Plyushcheva, Ukr. matem. zhurn., No. 1 (1962).
3. I. S. Iokhvidov, M. G. Krein, Tr. Mosk. matem. obshch., **8**, 413 (1958).
4. M. G. Krein, DAN, **53**, No. 1 (1946).
5. Yu. M. Berezanskii, Matem. sborn., **47** (89), 2 (1959).
6. I. S. Iokhvidov, M. G. Krein, Tr. Mosk. matem. obshch., **5**, 367 (1956).
7. M. G. Krein, Ukr. matem. zhurn., **1**, No. 4 (1949).

8. L. S. Pontryagin, Izv. AN SSSR, ser. matem., **8**, 243 (1944).

9. K. Maurin, Bull. Acad. Polon. Sci., Ser. math., **6**, No. 3, 149 (1958).

10. A. Devinatz, Trans. Am. Math. Soc., **77**, 455 (1954).

11. M. G. Krein, Sborn. tr. Inst. matem. AN USSR, **10**, 83 (1948).

Note: Figure translations are in progress. See original paper for figures.