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Abstract

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ESTIMATION OF THE INFLUENCE OF TURBULENT DISPERSION IN THE VERTICAL AND IN THE WIND DIRECTION ON THE PROPAGATION OF A POLYDISPERSE ADMIXTURE

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In works ([1], [2]), the propagation was considered of an admixture nonuniform with respect to the fall velocities of its particles, emitted by an instantaneous point source located at height $z = H$ above the Earth' s surface ($z = 0$). The concentration of the admixture as a function of spatial coordinates and time was determined by means of the solution of the equation of the semiempirical theory of turbulent diffusion with constant coefficients K_x, K_y, and K_z. The distribution density of the admixture substance over particle fall velocities w in the source was prescribed in the form of the two-parameter function*

$$N(w) = \frac{a^{n+1}}{\Gamma(n+1)} w^n e^{-aw}, \tag{1}$$

where $n > -1$.

The difference in the fall velocities of the admixture particles leads to its dispersion in the vertical, i.e., it produces the same effect as vertical turbulence. In work ([2]) it is shown that, during the deposition of a sufficiently nonuniform admixture (the parameters a and $|n|$ are sufficiently small), comparison of these two effects makes it possible to single out such a time interval $t_1(\varepsilon) < t < t_2(\varepsilon)$, within which, with an error of order ε, the turbulent dispersion of the admixture in the vertical with coefficient K_z may be neglected. In correspondence with the indicated time interval (t_1, t_2), on the Earth' s surface one may single out a region S, within which the concentration of the deposited polydisperse admixture may be calculated without taking account of vertical diffusion, with the same error of order ε. In ([2]) the region S was obtained for constant K_x and K_y, identical for all admixture particles. In this note we consider the case in which the horizontal dispersions $\sigma_x = \sqrt{2K_x t}$ and $\sigma_y = \sqrt{2K_y t}$ are proportional to time t (i.e., when the coefficients K_x and K_y are also proportional to t), while the turbulent dispersion of a polydisperse admixture in the vertical is small in comparison with the vertical dispersion caused by the difference in the fall velocities of particles of different weight fractions of the admixture. Thus the dependence of the turbulent dispersion coefficients of individual weight fractions on the fall velocity of their particles w will be taken into account (see ([3–5])), since, for $K_z = 0$, $t = (H - z)/w$, and the quantities K_x and K_y may be represented in the form ([1])

$$2K_x = \alpha \overline{U}_z^{\,2} \frac{H - z}{w} = \alpha \overline{U}_z^{\,2} t; \qquad 2K_y = \beta \overline{U}_z^{\,2} \frac{H - z}{w} = \beta \overline{U}_z^{\,2} t, \tag{2}$$

where

$$\overline{U}_z = \frac{1}{H - z} \int_z^H u(\zeta)\, d\zeta$$

is the mean wind speed in the layer $z \leq \zeta \leq H$.

To determine the region S in the case of coefficients K_x and K_y linearly dependent on t, we substitute (2) into the expression for the flux*** (for small K_z)

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* In ([1], [2]) it was assumed that $a = n/w_m,\ n > 0$.

** For estimates one may use values of K_z between 10^1 and 10^2 m^2/sec.

*** Dimension: unit mass/(unit area × unit time).

polydisperse admixture settling onto the Earth' s surface (formula (14) from ([2])), which in this case takes the form

$$\Pi(x, y) = \frac{Q(aH)^{n+1} e^{-1/2\alpha}}{2\pi\Gamma(n+1) u^2 \sqrt{\alpha\beta}} t^{-(n+4)} e^{b/t - r^2/t^2} \left\{ 1 + \frac{a^2}{t} K_z - \frac{2anK_z}{H} + \frac{n(n-1)}{H^2 \nu^2} K_{zt} + O\left[\left(\frac{a}{H} K_z \right)^2 \right] + O\left(e^{-H^2 \ldots} \right. \tag{3}$$

Here Q is the total amount of substance in the source; the horizontal x-axis is oriented in the direction of the mean wind velocity,

$$u = \bar{U}_0 = \frac{1}{H} \int_0^H u(\zeta)\, d\zeta,$$

$$b = \frac{x}{\alpha u} - aH; \qquad r^2 = \frac{x^2}{2\alpha u^2} + \frac{y^2}{2\beta u^2}; \qquad \nu = 1 - 2\frac{a}{H} K_z.$$

At the same time, evidently, the time interval (t_1, t_2) remains the same (see formula (15) from ([2])):

$$\frac{3}{\varepsilon}a^2K_z = t_1(\varepsilon) < t < t_2(\varepsilon) = \frac{H^2\nu^2}{K_z}\min\left\{\frac{\varepsilon}{3|n(n-1)|}, \frac{1}{8\left|\ln\frac{\varepsilon}{3}\right|}\right\}, \qquad (4)$$

where

$$\varepsilon > \frac{6aK_z}{H}\max\{1, |n|\}.$$

Integrating (3) with respect to time from 0 to ∞, we obtain an expression (for small K_z) for the concentration of the polydisperse admixture deposited on the Earth' s surface ($z = 0$)*:

$$p(x,y) = \frac{Q(aH)^{n+1}}{2\pi\Gamma(n+1)u^2H^2\sqrt{\alpha\beta}}\, e^{-1/2\alpha+b^2/4r^2}\frac{\sqrt{\pi}}{2}\Gamma(n+3)r^{-(n+3)}i_{n+2}\left(-\frac{b}{2r}\right)\times$$

$$\times\left\{1+\frac{a^2}{r}K_z\frac{\Gamma(n+4)}{\Gamma(n+3)}\frac{i_{n+3}(-b/2r)}{i_{n+2}(-b/2r)} - 2\frac{an}{H}K_z + \frac{n(n-1)}{H^2\nu^2}K_{zr}\frac{\Gamma(n+2)}{\Gamma(n+3)}\times\right.$$

$$\left.\times\frac{i_{n+1}(-b/2r)}{i_{n+2}(-b/2r)} + O\left[\left(\frac{a}{H}K_z\right)^2\right] + e^{[(s-b)^2-b^2]/4r^2}O\left[i_N\left(\frac{s-b}{2r}\right)\right]\Big/ i_{n+2}\left(-\frac{b}{2r}\right)\right\}, \qquad (5)$$

where

$$s = \frac{H^2\nu^2}{8K_z}; \qquad N < n+2; \qquad i_n(-z) = i^n\,\mathrm{erfc}(-z) = \frac{2}{\sqrt{\pi}}e^{-z^2}\frac{1}{\Gamma(n+1)}\int_0^\infty \tau^n e^{2z\tau-\tau^2}\,d\tau\ (^6).$$

The magnitude of the coefficients α and β is of order 10^{-3}, in connection with which the argument of the integral error function $i_{n+2}(-b/2r)$,

$$\frac{b}{2r} = \frac{1}{\sqrt{2\alpha}}\,\frac{1-\alpha\frac{au}{x}H}{\sqrt{1+\alpha y^2/\beta x^2}},$$

may be regarded as a large quantity when

$$\frac{\alpha y^2}{\beta x^2} < B < \infty, \qquad \alpha\frac{au}{x}H < \delta < 1. \qquad (6)$$

Then, using the asymptotic expansion for $i_n(-z)$ for large z (^{6}):

$$i_n(-b/2r) = \frac{2}{\Gamma(n+1)}\left(\frac{b}{2r}\right)^n \left\{1 + \frac{n(n-1)}{1!(b/r)^2} + \frac{n(n-1)(n-2)(n-3)}{2!(b/r)^4} + \cdots\right\},$$

we rewrite (5) in the form

$$p(x,y) = \frac{QN\left(\lambda H \frac{u}{x}\right)Hu}{\sqrt{2\pi\beta/\lambda}\, x\,(x/\lambda)^2} e^{-y^2\lambda^2/2\beta x^2} G(\alpha, \beta, K_z, x, y, a, n, H, u), \qquad (7)$$

* An expression for the principal term of this expansion in a somewhat different form was obtained in ([1]).

where

$$G = \mu^{n+2} e^{\frac{\alpha\lambda}{2}} \left(aH\frac{u}{x}\right)^2 \left\{1 + \frac{(n+2)(n+1)}{1!}\frac{\alpha}{2\lambda\mu^2} + O\left(\frac{\alpha^2}{4\lambda^2\mu^4}\right)\right\} \times$$
$$\times \left\{1 + a^2 K_z \frac{u}{x}\lambda\mu - \frac{2an}{H}K_z + \frac{n(n-1)}{H^2v^2}K_z\frac{x}{u\lambda\mu} + \right.$$
$$\left. + e^{\lambda s\frac{u}{x}\left[\frac{\alpha}{2}s\frac{u}{x} - \mu\right]} O\left\{ i_N\left[\sqrt{\frac{\lambda}{2a}}\left(\alpha s\frac{u}{x} - \mu\right)\right]\right\} / i_{n+2}\left[-\sqrt{\frac{\lambda}{2a}}\mu\right]\right\};$$
$$1-\delta < \mu = 1 - \alpha a H\frac{u}{x} < 1; \qquad \frac{1}{1+B} < \lambda = \frac{1}{1+\frac{\alpha y^2}{\beta x^2}} < 1;$$
$$\frac{b}{2r} = \mu\sqrt{\frac{\lambda}{2\alpha}}.$$

Obviously, if (3) is integrated with respect to y (the case of an infinite line source oriented across the wind), then in an expression analogous to (7) the principal term will have the form $QN(Hu/x)Hux^{-2}$.

Estimating the first-order terms in α, one can show that G will differ from unity by a quantity of order 2ε for

$$x_1(\varepsilon) < x < \frac{ut_2^2(\varepsilon)}{2\left(1+\frac{\alpha}{\beta}\frac{y^2}{x^2}\right)}; \qquad y^2 < x^2\frac{\beta}{\alpha}B, \qquad (8)$$

where

$$x_1(\varepsilon) = u\max\left\{\frac{\alpha}{1-\delta}\frac{H^2v^2}{8K_z}; \sqrt{\frac{3\alpha}{2\varepsilon}}\,aHu,\ t_1(\varepsilon),\ \frac{3(n+2)}{\varepsilon}\alpha a Hu,\ \frac{\alpha}{\delta}aHu\right\},$$

and the estimates ε, δ, and B are related by the inequality

$$1 + B < \frac{2}{3}\frac{\varepsilon(1-\delta)^2}{\alpha(n+2)(n+1)}.$$

Thus, the region S will be bounded by the straight line $x = x_1(\varepsilon)$, by the ellipse

$$\left(\frac{x - ut_2^2/4}{ut_2^2/4}\right)^2 + \frac{\alpha}{\beta}\left(\frac{y}{ut_2^2/4}\right)^2 = 1$$

and will lie inside the angle $|y| = x\sqrt{\frac{\beta}{\alpha}B}$.

Using the principal term of formula (7), in the region S one can solve the inverse problem, namely: from the distribution on the axis of the surface concentration $p(x,0)$* (the maximum of $p(x,0)$ is located at the point $x_{\max} = \frac{a}{n+3}Hu$) find the function $N(\mathfrak{w})$:

$$N(\mathfrak{w}) = N\left(H\frac{u}{x}\right) = \frac{p(x,0)\,\sigma_y\sqrt{2\pi x^2}}{QHu}, \tag{9}$$

where $\sigma_y = x\sqrt{\beta}$ is the dispersion of the distribution $p(x,y)$ in the direction of the y-axis.

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CITED LITERATURE

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* $p(x,0)$ does not depend on α, since on the axis $y = 0$, $\lambda = 1$.

Note: Figure translations are in progress. See original paper for figures.