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Abstract

Full Text

MATHEMATICS

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ON A BOUNDARY-VALUE PROBLEM FOR A SYSTEM OF TWO DIFFERENTIAL EQUATIONS

(Presented by Academician S. L. Sobolev, January 15, 1962)

The present note is devoted to the study of solutions $z(t) = \{x(t), y(t)\}$ of the system of differential equations

$$\frac{dx}{dt} = f(t, x, y),$$

$$\frac{dy}{dt} = g(t, x, y), \tag{1}$$

satisfying the boundary conditions

$$x(a)\sin\alpha - y(a)\cos\alpha = 0, \qquad x(b)\sin\beta - y(b)\cos\beta = 0, \tag{2}$$

$$(0 \le \alpha < \pi,\ 0 < \beta \le \pi).$$

Everywhere in what follows it is assumed that the right-hand sides of system (1) are continuous in the domain $D:\ a \le t \le b,\ -\infty < x, y < +\infty$ and ensure the continuability of any solution of system (1) over the whole interval $a \le t \le b$. In addition, it is assumed that the solution of system (1) is uniquely determined by the initial conditions. We note that a number of the theorems stated below are also valid without this assumption.

In the proof of the theorems an essential role is played by the passage to a polar coordinate system and the systematic use of differential inequalities ([3]).

Let $z(t) = \{x(t), y(t)\}$ be a solution of system (1) which does not vanish on the interval $a \le t \le b$. The angular function $\varphi(t)$ of the solution $z(t)$ is called the polar angle of the vector $z(t)$.

1. Comparison theorems. Along with system (1), consider the system

$$\frac{dx}{dt} = \tilde{f}(t, x, y),$$

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$$\frac{dy}{dt} = \tilde{g}(t, x, y). \tag{3}$$

Let $D^* \subseteq D$. We shall write $\{f, g\} \geq \{\tilde{f}, \tilde{g}\}$ in the domain D^* if the inequality

$$xg(t, x, y) - yf(t, x, y) \geq x\tilde{g}(t, x, y) - y\tilde{f}(t, x, y) \quad ((t, x, y) \in D^*) \tag{4}$$

is satisfied.

Let the solutions $z(t)$ and $\tilde{z}(t)$, respectively, of systems (1) and (3), be defined and nonzero on the whole interval $a \leq t \leq b$, with $(t, x(t), y(t)) \in D^*$, $(t, \tilde{x}(t), \tilde{y}(t)) \in D^*$. Denote their angular functions by $\varphi(t)$ and $\tilde{\varphi}(t)$.

Theorem 1. Let the right-hand sides of system (3) possess the property of positive homogeneity:

$$\tilde{f}(t, cx, cy) \equiv c\tilde{f}(t, x, y), \qquad \tilde{g}(t, cx, cy) \equiv c\tilde{g}(t, x, y) \quad (c \geq 0). \tag{5}$$

Let $\varphi(a) \geq \tilde{\varphi}(a)$, $\{f, g\} \geq \{\tilde{f}, \tilde{g}\}$ in the domain D^*.

Then $\varphi(t) \geq \tilde{\varphi}(t)$ $(a \leq t \leq b)$.

This theorem is a generalization of one result from [1]. Now let us proceed to a comparison of the angular functions of two solutions of system (1). We shall say that system (1) satisfies the (+)-condition if the expression $xg(t, x, y) - yf(t, x, y)$ $(a \leq t \leq b,\ -\infty < x, y < +\infty)$ is nonnegative and if the trajectories of two distinct solutions $z_1(t)$ and $z_2(t)$, satisfying the conditions

$$\arg z_1(a) = \arg z_2(a) = \alpha^*, \qquad \arg z_1(\tilde{b}) = \arg z_2(\tilde{b}) = \beta^*,$$

$$\alpha^* \leq \arg z_1(t), \arg z_2(t) \leq \beta^* \qquad (a \leq t \leq \tilde{b}),$$

do not intersect. The last condition is always fulfilled for autonomous systems, and also for systems whose right-hand sides satisfy the condition of positive homogeneity (4). The (−)-condition is introduced analogously.

By $\varphi(t, \alpha, \rho)$ we denote the angular function of the solution of system (1) satisfying the initial condition

$$x(a) = \rho \cos \alpha, \qquad y(a) = \rho \sin \alpha.$$

Theorem 2. *Let system (1) satisfy the (+)-condition or the (−)-condition. Suppose, moreover, that the function*

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$$F(t, \varphi, r) = \frac{1}{r}\left[g(t, r\cos\varphi, r\sin\varphi)\cos\varphi - f(t, r\cos\varphi, r\sin\varphi)\sin\varphi\right] \qquad (6)$$

is nondecreasing in r.

Then the function $\varphi(t, \alpha, \rho)$ *is nondecreasing in* ρ.

If the right-hand sides of system (1) are continuously differentiable with respect to the variables x and y, then the nondecrease of the function $F(t, \varphi, r)$ in r is equivalent to the nonnegativity of the derivative $F'_r(t, \varphi, r)$. It can be shown that the latter means the following:

$$g'_x x^2 + [g'_y - f'_x]xy - f'_y y^2 \geq gx - fy.$$

We note that Theorem 2 remains valid if, in its formulation, the words "does not decrease" are replaced by the words "does not increase."

2. The classes H_k. Suppose that the right-hand sides of system (1) satisfy condition (5). Denote by $\varphi^+(t)$ and $\varphi^-(t)$ the angular functions of the solutions of the system satisfying the initial conditions $\varphi^+(a) = \alpha$, $\varphi^-(a) = \alpha + \pi$. We shall call the boundary-value problem (1)–(2) **regular** if

$$\beta + k\pi < \varphi^+(b), \qquad \varphi^-(b) - \pi < \beta + (k+1)\pi \qquad (7)$$

for some integer k. In what follows the boundary conditions will be regarded as fixed; therefore one may speak of regularity of the right-hand side of system (1). By H_k we denote all regular right-hand sides $\{f, g\}$ for which inequality (7) is fulfilled.

From Theorem 1 there follows the following property of the classes H_k: if $\{f_+, g_+\} \gg \{f, g\} \gg \{f_-, g_-\}$ in D and $\{f_+, g_+\}, \{f_-, g_-\} \in H_k$, then $\{f, g\} \in H_k$.

3. Existence theorems. Suppose that the right-hand sides $f(t, x, y)$ and $g(t, x, y)$ satisfy the inequality

$$\underline{b}(t, x, y)x - \underline{a}(t, x, y)y + \underline{\delta}(t, x, y) \leq g(t, x, y)x - f(t, x, y)y \leq$$

$$\leq \overline{b}(t, x, y)x - \overline{a}(t, x, y)y + \overline{\delta}(t, x, y), \qquad (8)$$

where $\{\underline{a}(t, x, y), \underline{b}(t, x, y)\}$ and $\{\overline{a}(t, x, y), \overline{b}(t, x, y)\}$ belong to the same class H_k, while the functions $\underline{\delta}(t, x, y)$ and $\overline{\delta}(t, x, y)$ satisfy the conditions

$$\lim_{x^2+y^2\to+\infty} \frac{\underline{\delta}(t, x, y)}{x^2+y^2} = 0, \qquad \lim_{x^2+y^2\to+\infty} \frac{\overline{\delta}(t, x, y)}{x^2+y^2} = 0. \qquad (9)$$

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When the assumptions made above are fulfilled, we shall say that the right-hand side $\{f, g\}$ of system (1) satisfies the infinity k-condition (with respect to the boundary conditions (2)).

Theorem 3. *Let $\{f, g\}$ satisfy the k-condition at infinity. Then the boundary-value problem (1)–(2) is solvable.*

Theorem 3 is a generalization of one theorem from ([2]), proved for equations of the second order. It is not difficult to see that the assertion of the theorem remains valid if it is known that the inequality

$$\beta + k\pi < \varphi(b, \alpha, \rho), \qquad \varphi(b, \alpha + \pi, \rho) - \pi < \beta + (k+1)\pi \quad (\rho \geqslant \rho_0) \tag{10}$$

is satisfied. (We note that, in the case when the k-condition at infinity is satisfied, inequality (10) is valid; the converse is false.)

Let $z(t, c)$ be the solution of system (1) satisfying the initial condition $x(a) = c\cos\varphi,\ y(a) = c\sin\alpha$. Theorem 3 guarantees the existence of a solution of the boundary-value problem (1)–(2), since the curve $z(b, c)$ $(-\infty < c < +\infty)$, for sufficiently large values of $|c|$, lies in two sectors situated on different sides of the straight line $x\sin\beta - y\cos\beta = 0$.

However, it is geometrically clear that the boundary-value problem will also be solvable if the curve $z(b, c)$ has a spiral-like character either as c approaches zero (in the case when the boundary-value problem already has a zero solution) or as $|c|$ tends to infinity. Along this path one can obtain various sufficient criteria for solvability. One of them—in the form of a theorem—we give here.

Theorem 4. *Let the functions $f(t, x, y)$ and $g(t, x, y)$ be such that the inequality*

$$F(t, \varphi, r) \geqslant \varepsilon_0 > 0 \quad (a \leqslant t \leqslant b,\ 0 \leqslant \varphi \leqslant 2\pi) \tag{11}$$

is satisfied for $r \geqslant \rho_0$, and suppose that the plane (x, y) can be divided into a finite number of sectors $S_1, \dots, S_m$,

$$S_j = \{(\varphi, r):\ \varphi_j \leqslant \varphi \leqslant \varphi_{j+1},\ 0 \leqslant r\} \quad (\alpha \leqslant \varphi_1 \dots < \varphi_{m+1} = \varphi_1 + 2\pi),$$

so that

$$\lim_{r \to +\infty} F(t, \varphi, r) = +\infty \tag{12}$$

uniformly in all (t, φ), $a \leqslant t \leqslant b$, $\varphi_j + \varepsilon \leqslant \varphi \leqslant \varphi_{j+1} - \varepsilon$ $(j = 1, \dots, m)$, where the positive ε may be taken arbitrarily small.

Then the boundary-value problem (1)–(2) has an infinite number of solutions.

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4. Estimate of the number of solutions of the boundary-value problem. First of all we formulate the following uniqueness criterion.

Theorem 5. *Let the right-hand sides be continuously differentiable with respect to the variables x and y, let $f'_x(t,x,y)$ and $g'_y(t,x,y)$ be bounded, and suppose that the inequality*

$$\underline{A}(t) \leqslant \begin{pmatrix} f'_x(t,x,y) & f'_y(t,x,y) \\ g'_x(t,x,y) & g'_y(t,x,y) \end{pmatrix} \leqslant \overline{A}(t), \tag{13}$$

is satisfied, where $\underline{A}(t)$, $\overline{A}(t)$ are matrices of one class H_k.

Then the boundary-value problem has a unique solution.

From this theorem there follows a number of known uniqueness conditions ([1]).

In many cases it is of interest to determine when the boundary-value problem has several solutions, and to give an estimate of the number of such solutions.

We first give the following definition. Suppose that system (1) has the zero solution, i.e.

$$f(t,0,0) \equiv g(t,0,0) \equiv 0.$$

Suppose further that the functions $f(t,x,y)$ and $g(t,x,y)$ are continuously differentiable with respect to the variables x and y for $a \leqslant t \leqslant b$, $x^2 + y^2 \leqslant \rho_0^2$. Form the matrix

$$A(t) = \begin{pmatrix} f'_x(t,0,0) & f'_y(t,0,0) \\ g'_x(t,0,0) & g'_y(t,0,0) \end{pmatrix}.$$

If $A(t) \in H_l$, then we shall say that system (1) **satisfies the zero l-condition** (with respect to the boundary conditions (2)).

Theorem 6. Let system (1) satisfy the k-condition at infinity and the l-condition at zero, where $k \neq l$.

Then the boundary-value problem (1)–(2) has at least $2|k-l|$ zero solutions.

We note that if, under the hypotheses of Theorem 6, the hypotheses of Theorem 2 are satisfied, then the boundary-value problem (1)–(2) has exactly $2|k-l|$ nonzero solutions.

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