SOME CASES OF THE BIRTH OF PERIODIC MOTIONS IN n-DIMENSIONAL SPACE

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Abstract

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MATHEMATICS

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SOME CASES OF THE BIRTH OF PERIODIC MOTIONS IN n-DIMENSIONAL SPACE

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1. One of the basic problems of the qualitative theory of differential equations is the consideration of bifurcations of states of equilibrium and periodic motions ([1–6]). In the present note, conditions are established for the birth of stable periodic motions from trajectories going from a state of equilibrium back into it. The results obtained are a generalization of the corresponding results of A. A. Andronov and E. A. Leontovich ([2,3]) to the case of n-dimensional space.

2. Consider a system of n differential equations

$$\frac{dx_i}{dt} = X_i(x_1, x_2, \dots, x_n; \mu), \qquad i = 1, \dots, n, \tag{1}$$

depending on the parameter μ ($|\mu| < \mu_0$). With respect to the right-hand sides, assume that they are sufficiently smooth functions of the variables $x_1, \dots, x_n$ and μ. Suppose that for $\mu = 0$ system (1) has an equilibrium state $O(0, \dots, 0)$, in which the roots $\lambda_1, \dots, \lambda_{n-1}$ of the characteristic equation

$$\left| \left(\frac{\partial X_i}{\partial x_j} \right)_0 - \delta_{ij}\lambda \right| = 0 \tag{2}$$

have real parts of one sign, and $\lambda_n = 0$. By means of a nonsingular linear change, system (1) can be reduced to the form

$$\begin{aligned} \frac{dx_i}{dt} &= \sum_{j=1}^{n-1} a_{ij}x_j + P_i(x_1, \dots, x_n; \mu), \qquad i = 1, \dots, n-1; \\ \frac{dx_n}{dt} &= P_n(x_1, x_2, \dots, x_n; \mu), \end{aligned} \tag{3}$$

where the functions $P_i(x_1, \dots, x_n; 0)$, $i = 1, 2, \dots, n$, contain no linear terms with respect to $x_1, \dots, x_n$.

Consider the function

$$\overline{P}_n(x_n;\mu) = P_n(U_1 + \mu u_1, \dots, U_{n-1} + \mu u_{n-1}, x_n; \mu),$$

where

$$x_i = U_i(x_n) + \mu u_i(x_n;\mu), \qquad i = 1, \dots, n-1,$$

satisfy the system of equations

$$\sum a_{ij}x_j + P_i(x_1, \dots, x_n; \mu) = 0, \qquad i = 1, \dots, n-1.$$

Suppose that in the expansion $\overline{P}_n(x_n;0) = l_m x_n^m + \dots$ the first nonzero quantity l_m ([7]) has an even index and is positive (if

$l_m < 0$, then, replacing x_n by $-x_n$, we arrive at the case under consideration). In this case the equilibrium state O is called a saddle-node.

Let, for definiteness, $\operatorname{Re}\lambda_i < 0,\ i = 1, 2, \dots, n-1$. The behavior of the integral curves of system (3) for $\mu = 0$ in a sufficiently small neighborhood of O is described by the following lemma, which is a generalization of a result of R. M. Mintz ([8]), established for $n = 3$.

Lemma. *There exists an $(n-1)$-dimensional separatrix surface I_{n-1}, tangent to the plane $x_n = 0$, consisting of O^+-curves and dividing a neighborhood of O into nodal and saddle regions. In the nodal region every integral curve is an O^+-curve, while in the saddle region all integral curves, with the exception of one O^--curve, pass at a finite distance from the saddle-node.*

The condition that system (3) have no equilibrium states in a sufficiently small neighborhood of the origin is equivalent to the definiteness and positivity of the expression

$$\bar{P}_n(x_n;\mu) = l_m x_n^m(1 + q_1(x)) + \mu q_2(x_n;\mu).$$

In particular, if $q_2(0,0) > 0$, then as the parameter μ is varied the following bifurcation picture occurs: two simple equilibrium states—a saddle and a node, existing for $\mu < 0$, approach each other as μ increases and then merge into one equilibrium state (a saddle-node), which disappears for $\mu > 0$. Studying the equation $\bar{P}_n(x_n;\mu) = 0$, one can show that sufficiently small values of the parameter μ for which, in some fixed neighborhood of the origin, equilibrium states will be absent form intervals (one or two), one of whose endpoints will be the point with coordinate $\mu = 0$, corresponding to the existence of the saddle-node.* We denote the values of the parameter μ from the indicated intervals by $\bar{\mu}$.

According to Lemma 1, from the saddle-node there issues only one O^--curve. Denote it by Γ_0.

Theorem 1. *If the trajectory Γ_0 as $t \to +\infty$ enters the saddle-node and does not lie on the separatrix surface, then, when the saddle-node disappears (as a result of a change of the parameter), from Γ_0 there is born only one periodic motion, stable for $\operatorname{Re}\lambda_i < 0,\ i = 1, \dots, n-1$.*

The proof of the theorem is based on the method of point transformations.

Denote by S_0 and S_1 the surfaces $x_n = -d$ and $x_n = d$, where $d > 0$. For sufficiently small d, the trajectory Γ_0 will intersect S_0 and S_1 at the points M_0^0 and M_1^0. One can show that for all sufficiently small $\bar{\mu}$ (in absolute value) on the surface S_0 there is a σ_0-neighborhood of the point M_0^0 such that the trajectory passing at $t = 0$ through a point $M^0 \in \sigma_0$ intersects, for some $t = t_1$, the surface S_1 at a certain point M^1. The point transformation, thus established, of the domain σ_0 onto the surface S_1 will be denoted by T_0. It has also been proved that as $\bar{\mu} \to 0$ the domain $T_0\sigma_0$ is uniformly contracted to the point M_1^0, and the norm of the matrix of the linearized transformation T_0 tends to zero. Since Γ_0 intersects the surfaces S_1 and S_0 without tangency, it follows from the general theorems of the theory of point transformations ([6]) that for every point sufficiently close to the point M_1^0 one can indicate a certain point on the surface S_0. Moreover, one can indicate such a σ_1-neighborhood of the point M_1^0 on S_1 that its image under the indicated correspondence, which we denote by T_1, will lie inside σ_0. Consider the transformation $T = T_1 T_0$. The transformation T is contracting for all sufficiently small $\bar{\mu}$. From the contractiveness of T and the known relation ([9]) between periodic motions of the dynamical—

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* We note that the existence of two intervals $(-\mu^*, 0)$ and $(0, \mu^*)$ is possible in the case when $x_n = \mu = 0$ is an isolated solution of the equation $\bar{P}_n(x_n; \mu) = 0$.

mical system and fixed points of the corresponding mapping, there follows the existence of a unique and stable periodic motion.

3. Let a system of the form (1), for $\mu = 0$, have a simple equilibrium state of saddle type, in which the roots $\lambda_1, \dots, \lambda_{n-1}$ of the characteristic equation are distinct and have negative parts, while $\lambda_n > 0$. With respect to the functions X_i we shall assume that they are twice continuously differentiable. By a change of variables, the system under consideration, for sufficiently small μ, can be brought to the form

$$\begin{aligned} \frac{dx_i}{dt} &= \sum_{j=1}^{n-1} a_{ij}(\mu) x_j + P_i(x_1, \dots, x_n; \mu), \qquad i = 1, \dots, n-1; \\ \frac{dx_n}{dt} &= \lambda_n(\mu) x_n + P_n(x_1, \dots, x_n; \mu), \end{aligned} \tag{4}$$

where the functions $P_i(x_1, \dots, x_n; \mu)$, $i = 1, \dots, n$, contain no linear terms in $x_1, \dots, x_n$ and, moreover,

$$P_i(0, \dots, 0; \mu) \equiv 0.$$

As is known ([7, 10–12]), in this case there exist two integral manifolds I_1 and I_{n-1}, on which all the 0^--curves of the equilibrium state under consideration are situated.

The equations of the manifold I_1, consisting of two 0^--curves, are written in the form

$$x_i = F_i(x_n; \mu_1), \qquad i = 1, \dots, n-1,$$

where $\partial F_i(0; \mu)/\partial x_n = 0$, $i = 1, \dots, n-1$, and the equation of the manifold I_{n-1}, consisting of 0^+-curves, is in the form

$$x_n = F_n(x_1, \dots, x_{n-1}; \mu),$$

where $\partial F_n(0, \dots, 0; \mu)/\partial x_i = 0$, $i = 1, \dots, n-1$. The manifold I_{n-1} divides a sufficiently small neighborhood of the origin into two regions, which we denote by D^+ (D^+ contains a segment of the positive semiaxis Ox_n) and D^- (D^- contains a segment of the negative semiaxis Ox_n). Denote by Γ_0 the trajectory issuing from the saddle into the region D^+.

Theorem 2. *If: 1) for $\mu = 0$ and $t \to +\infty$ the trajectory Γ_0 enters the saddle O, and for $\mu > 0$ ($\mu < 0$) passes by the saddle, crossing D^+; 2) $-\operatorname{Re}\lambda_i > \lambda_n$, $i = 1, \dots, n-1$, then, for sufficiently small $\mu > 0$ ($\mu < 0$), from Γ_0 there is born only one periodic motion, stable as $t \to +\infty$.*

One can specify a surface S_0 whose equation is $S_0(x_1, \dots, x_{n-1}) = 0$, where the function S_0 is twice continuously differentiable, such that Γ_0, for sufficiently small μ, will intersect S_0. Denote the point of intersection of Γ_0 with S_0 by M_μ^0, and by γ the intersection of the surfaces S_0 and I_{n-1}. Obviously, the point M_0^0 (the point M_μ^0 for $\mu = 0$) lies on γ. It can be proved that, for every sufficiently small and fixed μ, a trajectory of system (4) passing at $t = 0$ through a point M^0, close to $M_0^0 \in S_0$ and lying in D^+, for some $t = t_1$ again intersects the surface S_0. Denote the resulting mapping of the surface S_0 onto itself by T. If $-\operatorname{Re}\lambda_i > \lambda_n$, $i = 1, \dots, n-1$, then for sufficiently small μ, at points M^0 close to γ, the norm of the linearized mapping T will be less than the number q, where $q < 1$.

We note that the mapping T can be completed by continuity on the curve γ itself and then continued, with preservation of smoothness, to that part of the surface S_0 which lies in D^-. This extended mapping, which we denote by $\widetilde{T}$, for $\mu = 0$ will have a fixed point M_0^0. By the implicit function theorem, since the Jacobian at M_0^0 is less than uniboundary, it follows that for sufficiently small μ the mapping $\widetilde{T}$ will have a unique and stable fixed point M_μ^*, tending to M_0^0 as $\mu \to 0$. It can be proved that the point M_μ^* will lie in D^+ and, consequently, is a fixed point of the mapping T, if M_μ^0 also lies in D^+. Using the connection between periodic motions and fixed points, we consequently arrive at Theorem 2.

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Note: Figure translations are in progress. See original paper for figures.