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Abstract

Full Text

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SUMMATION OF FOURIER INTEGRALS AND BEST APPROXIMATION BY ENTIRE FUNCTIONS

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Let us consider the space $L_p(-\infty,\infty)$, $1 \le p \le \infty$, of functions $f(x)$ defined on the entire real axis, i.e.,

$$\|f(x)\|_{L_p} = \left\{ \int_{-\infty}^{\infty} |f(x)|^p dx \right\}^{1/p} < \infty \qquad (1 \le p < \infty),$$

$$\|f(x)\|_{L_\infty} = \operatorname{vrai\,sup}_{-\infty<x<\infty} |f(x)| < \infty.$$

Denote

$$U_\lambda(f;x;\varphi) = \int_{-\lambda}^{\lambda} \varphi_\lambda(u) F(u) e^{iux}\, du,$$

where $F(u)$ is the Fourier transform of the function $f(x)^*$, and $\varphi_\lambda(u)$ is a certain function satisfying the conditions:

$$\varphi_\lambda(u) = \begin{cases} 1, & \text{for } u = 0, \\ 0, & \text{for } |u| \ge \lambda, \end{cases}$$

$$\varphi_\lambda(-u) = \varphi_\lambda(u), \qquad \varphi''_\lambda(u) \in L_p(-\infty,\infty).$$

Let

$$R_\lambda(f;\varphi)_{L_p} = \|f(x) - U_\lambda(f;x;\varphi)\|_{L_p}$$

and let $A_\sigma(f)_{L_p}$ denote the best approximation of the function in the corresponding metric by entire functions of degree $\le \sigma$ belonging to the space $L_p(-\infty,\infty)$, i.e.,

$$A_\sigma(f)_{L_p} = \inf_{g_\sigma} \|f(x) - g_\sigma(x)\|_{L_p},$$

where $g_\sigma(x) \in L_p$ is an entire function of degree $\le \sigma$.

Below are given some results concerning the order of decrease of the quantity $R_\lambda(f;\varphi)_{L_p}$ $(\lambda \to \infty)$, depending on the behavior of the function $A_\sigma(t)_{L_p}$ (as $\sigma \to \infty$).

Theorem 1. Let $H_{\psi(\lambda)}$ denote the class of functions uniformly continuous on the entire axis, whose best approximation by entire functions of degree $\leq \lambda$ satisfies the inequality

$$A_\lambda(t) \leq \psi(\lambda),$$

where $\psi(\lambda)$ is a function continuous from the right and decreasing to zero as $\lambda \to \infty$. Then for $p = \infty$ the inequality

$$\sup_{f \in H_{\psi(\lambda)}} R_\lambda(t; \varphi)_{L_\infty} \geq \left| \int_0^\lambda \varphi'_\lambda(t) \psi(t)\, dt \right|. \tag{1}$$

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* Under the assumption that $F(u)$ exists and belongs to $L_q(-\infty, \infty)$, $1 \leq q \leq \infty$.

In particular, it follows from this, for example, that for

$$\varphi_\lambda(u) = 1 - \frac{|u|^r}{\lambda^r} \qquad (r \geqslant 1) \tag{2}$$

$$\sup_{f \in H_\psi(\lambda)} R_\lambda\left(f; 1 - \frac{|u|^r}{\lambda^r}\right)_{L_\infty} \geqslant \begin{cases} \left| \dfrac{r}{\lambda^r} \displaystyle\int_0^\lambda t^{r-1} \psi(t)\, dt \right|, & \text{for } r > 1, \\ \left| \dfrac{1}{\lambda} \displaystyle\int_0^\lambda \psi(t)\, dt \right|, & \text{for } r = 1. \end{cases} \tag{3}$$

The following theorem gives an upper estimate for the quantity $R_\lambda(f; \varphi)_{L_p}$.

Theorem 2. If $f(x) \in L_p(-\infty, \infty)$, $1 \leqslant p \leqslant \infty$, then

$$R_\lambda(f; \varphi)_{L_p} \leqslant C \left\{ |\varphi'_\lambda(0)| \int_0^\lambda A_u(f)_{L_p}\, du + \int_0^\lambda |\varphi''_\lambda(u)| A_u(f)_{L_p} (\lambda - u) \ln \frac{\lambda + u}{\lambda - u}\, du \right\}. \tag{4}$$

It follows directly from Theorem 2 that for the function $\varphi_\lambda(u)$ defined by equality (2), the estimate

$$R_\lambda\left(f; 1 - \frac{|u|^r}{\lambda^r}\right)_{L_p} \leqslant \frac{C_r}{\lambda^r} \int_0^\lambda u^{r-1} A_u(f)_{L_p}\, du. \tag{5}$$

Estimates (3) and (5) give the order relation

$$\sup_{f \in H_\psi(\lambda)} R_\lambda\left(f; 1 - \frac{|u|^r}{\lambda^r}\right)_{L_\infty} \asymp \frac{1}{\lambda^r} \int_0^\lambda u^{r-1} \psi(u)\, du. \tag{6}$$

Theorem 2 is proved with the aid of inverse theorems of the constructive theory of functions defined on the whole real axis (see ([4]), Theorem 2), the integral representation of Bochner-Fejér sums (see ([5])) and the known estimate ([2])

$$\frac{1}{\pi u}\int_{-\infty}^{\infty}\left|\frac{2\sin\frac{2\lambda-u}{2}t\,\sin\frac{u}{2}t}{t^2}\right|dt \leqslant \frac{\pi}{4}+\frac{2}{\pi}\ln\frac{2\lambda-u}{u}.$$

As relation (6) shows, inequality (5) for $p=\infty$ cannot be improved in order.

We give one assertion showing that in the case $1<p<\infty$ inequality (5) can be replaced by a sharper one.

Theorem 3. For any function $f(x)\in L_p(-\infty,\infty)$, with $1<p<\infty$, the inequality

$$R_\lambda\left(f;1-\frac{|u|^r}{\lambda^r}\right)_{L_p} \leqslant \frac{C_{p,r}}{\lambda^r}\left(\sum_{\nu=0}^{[\lambda]}(\nu+1)^{\gamma r-1}A_\nu^\gamma(f)_{L_p}\right)^{1/\gamma}, \tag{7}$$

holds, where $\gamma=p$ for $1<p\leqslant 2$ and $\gamma=2$ for $2\leqslant p<\infty$.

The proof of Theorem 3 is substantially based on the results of the work of M. F. Timan ([4]).

Using Parseval' s equality for $p=2$, it is not hard to verify that for every function $f(x)\in L_2(-\infty,\infty)$ the order relation

$$R_\lambda\left(f;1-\frac{|u|^r}{\lambda^r}\right)_{L_2} \asymp \frac{1}{\lambda^2}\left(\sum_{\nu=0}^{[\lambda]}(\nu+1)^{2r-1}A_\nu^2(f)_{L_2}\right)^{1/2}.$$

The method of proof of Theorem 2 makes it possible to obtain the following assertion for uniformly almost-periodic functions whose Fourier exponents have no finite limit points.

Theorem 4*. Let $f(x)$ be a uniformly almost-periodic function whose Fourier exponents have no finite limit points.

Then

$$\max_x|f(x)-\sigma_\lambda(f;x)|\le\frac{C}{\lambda}\int_0^\lambda A_u(f)\,du,$$

where

$$\sigma_\lambda(f;x) = \sum_{|\Lambda_k|<\lambda} \left(1 - \frac{|\Lambda_k|}{\lambda}\right) a_k e^{i\Lambda_k x};$$

a_k are the Fourier coefficients; Λ_k are the Fourier exponents of the function $f(x)$ ($\Lambda_0 = 0,\ \Lambda_{-k} = -\Lambda_k,\ \Lambda_k < \Lambda_{k+1}$ for $k = 0, 1, 2, \dots;\ \lim_{k\to\infty} \Lambda_k = \infty$).

In particular, when

$$A_u(f) = O\left[\frac{1}{(u+1)^\alpha}\right] \qquad (0 < \alpha \le 1),$$

we obtain the estimate

$$|f(x) - \sigma_\lambda(f;x)| \le \begin{cases} C\dfrac{1}{\lambda^\alpha}, & \text{for } 0 < \alpha < 1, \\ C'\dfrac{\ln\lambda}{\lambda}, & \text{for } \alpha = 1, \end{cases}$$

which constitutes the content of Theorems 2 and 3 of paper ([1]).

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CITED LITERATURE

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[2] B. M. Levitan, *Almost-Periodic Functions*, Moscow, 1953.
[3] S. B. Stechkin, *Tr. Mat. Inst. im. V. A. Steklova AN SSSR*, **62** (1961).
[4] M. F. Timan, *Izv. vyssh. uchebn. zaved.*, Mathematics, No. 6 (25), 108 (1961).
[5] E. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, Moscow-Leningrad, 1948.

* For the case of continuous 2π-periodic functions, an analogous theorem was obtained by different methods by S. B. Stechkin ([3]) and M. F. Timan. We note that M. F. Timan obtained a more general result, which he reported at the seminar of the Department of Higher Mathematics of the Dnepropetrovsk Agricultural Institute on 12 X 1961.

Note: Figure translations are in progress. See original paper for figures.