MATHEMATICS

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1962

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Abstract

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THE FOURIER TRANSFORM FOR ABSTRACT SET FUNCTIONS

(Presented by Academician S. L. Sobolev, 23 I 1962)

In the paper ([1]), S. L. Sobolev, in connection with embedding theorems for abstract functions, considered the B-spaces $\Phi_p(\Omega)$, $p \geqslant 1$, of abstract additive set functions $\varphi(E)$, defined for all L-measurable sets E from some bounded domain $\Omega \in R_n$. Here, for any L-measurable set $E \in R_n$, the function $\varphi(E)$ is defined by the equality $\varphi(E) = \varphi(E \cap \Omega)$. In the present note we construct the B-space $\Phi_p(R_n)$ of abstract additive set functions $\varphi(E)$, defined for all L-measurable sets $E \in R_n$ of finite measure (in the case $p = 1$ the measure of E may also be infinite), and then, for these spaces, as well as for certain subspaces $\Psi_1^*(R_n) \subset \Phi_1(R_n)$ and $\Psi_2^*(R_n) \subset \Phi_2(R_n)$, we consider the Fourier transform. The theorems proved are analogous to the Riemann–Lebesgue theorem on the Fourier transform for the space $L_1(R_n)$ and to Plancherel' s theorem on the Fourier transform for the space $L_2(R_n)$.

Let X be a B-space in which multiplication by complex numbers is defined. Let $\varphi(E)$ be an abstract additive function of L-measurable sets $E \in R_n$, taking, for each E, values in X, such that $\|\varphi(E)\|_X \leqslant A_E < \infty$. In particular, $\|\varphi(R_n)\|_X \leqslant A_{R_n} < \infty$.

By introducing the norm $\|\varphi(E)\|_M = \sup\limits_{E \in R_n} \|\varphi(E)\|_X$, from the vector manifold of such functions one can single out the B-space $M(R_n)$. The norm of $M(R_n)$ is equivalent to the norm
$\|\varphi\|_{\Phi_1(R_n)} = \sup\limits_{E_1 \cap E_2 = 0} \|\varphi(E_1) - \varphi(E_2)\|_X$.

The totality $\overline{\Phi}_1(R_n)$ of normal additive functions $\varphi(E)$ constitutes a subspace of the space $\Phi_1(R_n)$. Normality means the convergence to zero of the norm $\|\varphi(E_k)\|_X$ as $k \to \infty$, where $\{E_k\}$ is a vanishing sequence of sets.

In contrast to the space $\Phi_1(\Omega)$ of set functions from a bounded domain $\Omega \in R_n$, absolute continuity and even continuity with respect to translation of a function $\varphi(E) \in \Phi_1(R_n)$ do not imply the normality of $\varphi(E)$. Moreover, in contrast to the space of numerical functions $L_1(R_n)$, the totality of finite functions is dense not in all of $\Phi_1(R_n)$, but in the space $\Psi_1(R_n)$ of functions $\varphi(E) \in \Phi_1(R_n)$ continuous with respect to translation.

Theorem 1. *The totality of finite functions $\varphi(E) \in \Phi_1(R_n)$ is dense in $\overline{\Phi}_1(R_n)$.*

Considering integrals of the form

$$\int_{R_n} \omega(x)\, d\varphi(E),$$

where $\varphi(E) \in \Phi_1(R_n)$, and $\omega(x)$ is a bounded step function taking only a finite number of values, one can obtain the estimate

$$\left\| \int_{R_n} \omega(x)\, d\varphi(E) \right\|_X \leqslant 2 \max_x |\omega(x)|\, \|\varphi(E)\|_{\Phi_1(R_n)}.$$

Hence, by passage to the limit, the integral $\int_{R_n} \omega(x)\, d\varphi(E)$ is defined,

where $\omega(x)$ is an arbitrary bounded measurable complex-valued function of the real variable x.

Consider the space $\Psi_1^*(R_n) = \Phi_1^*(R_n) \cap \Psi_1(R_n)$, which is the closure, in the metric $\Phi_1(R_n)$, of the collection of abstract finite continuous point functions.

Theorem 2. To every function $\varphi(E) \in \Psi_1^*(R_n)$ one can assign its Fourier transform

$$F(u) = \frac{1}{(2\pi)^{n/2}} \int_{R_n} e^{i(u,x)}\, d_x\varphi(E).$$

The function $F(u)$ is uniformly continuous, $\|F(u)\|_X \to 0$ as $|u| \to \infty$.

Remark. The Fourier transform $F(u)$ can also be constructed for any function $\varphi(E) \in \Phi_1(R_n)$. However, it is easy to indicate an example of a function $\varphi(E) \in \Phi_1(R_n)$ whose Fourier transform is not a continuous function and does not tend to zero in norm as $|u| \to \infty$.

Alongside the space $\Phi_1(R_n)$, one may consider the spaces $\Phi_p(R_n)$, $p > 1$, of abstract additive functions $\varphi(E)$, defined on the collection of bounded L-measurable sets $E \in R_n$ with values in a B-space X with bounded norm

$$\|\varphi\|_{\Phi_p(R_n)} = \sup_{\omega \in L_{p'}(R_n)} \frac{\left\| \int_{R_n} \omega(x)\, d\varphi(E) \right\|_X}{\|\omega\|_{L_{p'}(R_n)}}, \qquad p > 1, \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

Let us dwell in more detail on the case $p = 2$. The space $\Phi_2(R_n)$ is a B-space. Every function $\varphi(E) \in \Phi_2(R_n)$ is absolutely continuous in the metric $\Phi_1(R_n)$, i.e. continuous in the norm X, and is defined not only for bounded, but also for unbounded L-measurable sets $E \in R_n$ of finite measure. The collection $\Phi_2^*(R_n)$ of finite functions from $\Phi_2(R_n)$ is dense neither in $\Phi_2(R_n)$ nor in $\Psi_2(R_n)$, where

$\Psi_2(R_n)$ is the space of functions from $\Phi_2(R_n)$ that are continuous with respect to shifts.

Denote by $\Psi_2^*(R_n)$ the intersection $\Phi_2^*(R_n) \cap \Psi_2(R_n)$. The space $\Psi_2^*(R_n)$ is the closure of the set of finite continuous abstract point functions in the metric $\Phi_2(R_n)$. There exist functions $\varphi(E) \in \Psi_2^*(R_n)$ that do not belong to $\Phi_1(R_n)$. On the other hand, there exist functions $\varphi(E) \in \Psi_1^*(R_n) \cap \Phi_2(R_n)$ that do not belong to $\Phi_2^*(R_n)$.

Finally, let us note one more feature of the space $\Phi_p(R_n)$, $p > 1$, distinguishing this space from the space $\Phi_p(\Omega)$. There exist abstract point functions $\varphi(x) \in \Phi_p(R_n)$, continuous in the norm X, which at the same time are not continuous with respect to shifts in the norm $\Phi_p(R_n)$.

Simple considerations show that the Fourier transform for a function $\varphi(E) \in \Phi_2(R_n)$ in the form of the integral

$$\int_{R_n} e^{-i(u,x)}\, d_x\varphi(E)$$

may have no meaning. At the same time, the following is valid:

Theorem 3. To every function $\varphi(E) \in \Phi_2(R_n)$ one can assign its Fourier transform

$$\Psi(E') = \frac{1}{(2\pi)^{n/2}} \int_{R_n} \left[\int_{E'} e^{i(u,x)}\, du\right] d_x\varphi(E)$$

so that $\Psi(E') \in \Phi_2(R_n)$, $\|\Psi\|_{\Phi_2(R_n)} = \|\varphi\|_{\Phi_2(R_n)}$, and

$$\varphi(E) = \frac{1}{(2\pi)^{n/2}} \int_{R_n} \left[\int_{E} e^{i(u,x)}\, dx\right] d_u\Psi(E').$$

If $\varphi(E) \in \Psi_2^*(R_n)$, then also $\Psi(E') \in \Psi_2^*(R_n)$.

As an example, let us construct the Fourier transform for the function

$$\frac{1}{\sqrt{2\pi}} e^{iux}.$$

1. We form the set function $\varphi(E) = \int_E \frac{1}{\sqrt{2\pi}} e^{iux}\, dx$. For any bounded $E \in R_1$, the function $\varphi(E)$ takes its value in $X = L_2(R_1)$. It is easy to see that $\varphi(E) \in \Phi_2(R_1)$. The Fourier transform for it is the abstract function $\Psi(E') = \chi_{E'}(x) \in \Phi_2(R_1)$; $\Psi(E') \in L_2(R_1)$ for each fixed E' of finite Lebesgue measure.

If the range of variation of u is a bounded domain $\Omega \in R_1$, then instead of a set function one may consider the point function

$$\frac{1}{\sqrt{2\pi}} e^{iux}.$$

2. Consider the space $\mathcal{L} = L_1(R_1) \cap L_2(R_1)$ with norm $\|f\|_L = \|f\|_{L_1} + \|f\|_{L_2}$, and let $X = \mathcal{L}^*$, where $\mathcal{L}^*$ is the space of linear functionals on $\mathcal{L}$. Then the function $\frac{1}{\sqrt{2\pi}} e^{iux}$ may be regarded as an abstract function of the point u with values in $X = \mathcal{L}^*$. In this case the functional is defined by the formula

$$l(u \mid f) = \frac{1}{\sqrt{2\pi}} \int_{R_n} e^{iux} f(x)\, dx.$$

The function $l(u)$ belongs to $\Psi_2^*(R_1)$. The Fourier transform for it is the abstract function $\Psi(E')$ with values in $X = \mathcal{L}^*$,

$$\Psi(E' \mid f) = \int_{E'} f(x)\, dx.$$

In conclusion, I express my gratitude to Academician S. L. Sobolev for his attention to this work.

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Received
17 I 1962

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Note: Figure translations are in progress. See original paper for figures.