MATHEMATICS

1962

SovietRxiv

Abstract

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MATHEMATICS

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ON SOME SYSTEMS OF DIFFERENTIAL EQUATIONS OF MIXED TYPE

(Presented by Academician I. N. Vekua, 15 I 1962)

Consider a system of partial differential equations of the second order in the plane of the variables (x, y):

$$\mathcal{L}u = G(y)u_{xx} - u_{yy} - K(y)u = h, \tag{1}$$

where $u = \{u_1, \dots, u_n\}$; $G(y)$, $K(y)$ are given symmetric square matrices satisfying the conditions: $K(y) \leqslant 0$, $K'_y(y) \geqslant 0$, $G(y) > 0$ for $y > 0$, $G(y) < 0$ for $y < 0$, $G'_y(y) > 0$. Matrix inequalities here and below are understood as inequalities for the corresponding quadratic forms constructed on arbitrary vectors of nonzero length.

In the upper half-plane the characteristic polynomial for system (1),

$$\det \left| G(y) - \lambda^2 E \right|$$

has n real roots $\lambda_1^2, \dots, \lambda_n^2$. Thus, from each point of the axis $y = 0$ into the half-plane $y > 0$ there issue two families of characteristics: n nonnegative ones (denote them by $\lambda_1^+, \dots, \lambda_n^+$) and n nonpositive ones (denote them by $\lambda_1^-, \dots, \lambda_n^-$).

In the lower half-plane the characteristic polynomial for system (1) has no real roots. By a mixed domain we shall mean a domain containing within itself an interval of the axis $y = 0$.

On the axis $y = 0$ take two arbitrary points $A(x_1, 0)$ and $B(x_2, 0)$; from them draw into the half-plane $y > 0$ twice continuously differentiable lines: one from the point A, so that along it

$$\frac{dx}{dy} \geqslant \sup_{\lambda_i^+}\{\lambda_1^+, \dots, \lambda_n^+\} = \Lambda^+(y),$$

the other—from the point B, so that along it

$$\frac{dx}{dy} \geqslant \sup_{\lambda_i^-}\{\lambda_1^-, \dots, \lambda_n^-\} = \Lambda^-(y).$$

We choose the directions of these lines so that they intersect. Denote their point of intersection by $C(x_0, y_0)$.

In the lower half-plane ($y < 0$) from the point A to the point B draw a twice continuously differentiable line σ so that on it

$$\left|\frac{dx}{dy}\right| > \sup_{0<y<y_0} \Lambda^+(y).$$

Let Ω be a simply connected mixed domain of the plane of the variables (x, y), bounded in the lower half-plane ($y < 0$) by the line σ with endpoints at A and B, and in the upper half-plane ($y > 0$) by the lines AC and CB.

The problem for system (1) in the domain Ω consists in finding a vector $u = \{u_1, \dots, u_n\}$, which is a solution of system (1) in the domain Ω and vanishes on the boundary $AC + \sigma$:

$$u\Big|_{AC+\sigma} = 0. \tag{2}$$

Let us denote by $\dot{C}^2$ the set of twice continuously differentiable vectors satisfying condition (2). The closure of this set in the norm

$$\|v\|_{W_2^s} = \int_\Omega \sum_{k=0}^{s} \left[D^k v \cdot D^k v\right]\, d\Omega,$$

where D^k denotes differentiation of order k, will be denoted by W_2^s. By W_2^0 we denote the Hilbert space of square-integrable vectors (the scalar product in it is $(v \cdot u) = \int_\Omega [v_1 u_1 + \dots + v_{nu} n]\, d\Omega$).

Following the arguments of ([4]), define a negative norm for each element $f \in W_2^0$ by the formula

$$\|f\|_{W_2^{-s}} = \sup_v \frac{(f \cdot v)}{\|v\|_{W_2^s}}.$$

The closure in this norm of the set W_2^0 will be denoted by W_2^{-s}. It is known ([4]) that any linear functional on W_2^{-s} can be represented in the form

$$(f \cdot v), \tag{3}$$

where $f \in W_2^{-s}$, and $v \in W_2^s$.

Lemma 1. *For vectors $u \in \dot{C}^2$ the inequality holds*

$$\|\mathcal{L}u\|_{W_2^0} \geq \text{const}\,\|u\|_{W_2^1}.$$

Lemma 1 permits one to draw the conclusion on the uniqueness of a smooth solution of the formulated problem for system (1).

If the adjoint homogeneous problem has only the zero solution, then the solvability of the posed problem in the sense of W_2^{-2} is obtained as in ([1]). In our case, however, solvability is obtained more simply by using the following lemma, which follows from (3) and Lemma 1.

Lemma 2. *The range of values of the operator $\mathcal{L}$, defined on $\dot{C}^2$, is dense in the space W_2^{-2}.*

Hence follows the solvability of the posed problem in the generalized sense: for any prescribed vector $h \in W_2^{-2}$ there exists a sequence of twice continuously differentiable vectors $\{u_n\}$, vanishing on the boundary $AC+\sigma$, such that $\mathcal{L}u_n$ tends to h in the metric of W_2^{-2}.

Consider the special case of system (1):

$$yu_{xx} - u_{yy} - K(y)u = h. \tag{4}$$

Let Ω be a simply connected mixed domain in the plane of variables (x, y), bounded in the upper half-plane $(y > 0)$ by the characteristics AC and BC of system (4), and in the lower half-plane $(y < 0)$ by a twice continuously differentiable arc σ:

$$\left|\frac{dx}{dy}\right| > \sqrt{-y_0}$$

(where y_0 is the ordinate of the point C), with endpoints at the points $A(x_1, 0)$ and $B(x_2, 0)$.

Let us write Green' s formula:

$$(v \cdot \mathcal{L}u) - (u \cdot \mathcal{L}^*v) = \int_\Gamma |M_n| \left\{\frac{\partial u}{\partial \mu} v - u\frac{\partial v}{\partial \mu}\right\} d\Gamma, \tag{5}$$

where $\Gamma = AC + CB + \sigma$; $M = \begin{pmatrix} y & 0 \\ 0 & -1 \end{pmatrix}$; $\mu = M_n/|M_n|$ is the unit normal; $n = \{n_x, n_y\}$ is the unit outer normal; $\mathcal{L}^*$ is the operator adjoint to the operator (4).

Formula (5) is valid for all vectors u and v which, say, have first derivatives continuous up to the boundary, and second derivatives continuous in the domain Ω. If the condition $nM_n = 0$ is fulfilled, then the unit conormal has a direction tangent to the boundary of the domain Ω; therefore the derivative $\partial u/\partial \mu$ is defined by specifying a smooth vector on this part of the boundary. In our case the equality $nM_n = 0$ holds along the characteristics in the upper half-plane.

From formula (5) it is easy to see that the mutually adjoint boundary conditions will be:

$$u|_{AC+\sigma} = 0, \tag{6}$$

$$v|_{BC+\sigma} = 0. \tag{7}$$

Lemma 3. *For smooth vectors u satisfying the boundary conditions (6) or (7), the inequality*

$$\|\mathcal{L}u\|_{W_2^0} \geq \text{const}\,\|u\|_{W_2^1}$$

holds.

Let $h \in W_2^0$; a vector $u \in W_2^0$ will be called a weak solution of problem T for the system (4), satisfying in the weak sense the boundary conditions (6), if the relation

$$(v \cdot h) - (u \cdot \mathcal{L}^* v) = 0$$

holds for all twice continuously differentiable vectors v satisfying the adjoint boundary conditions (7).

Let $h \in W_2^0$; a vector $u \in W_2^0$ will be called a strong solution of problem T for the system (4), satisfying the boundary conditions (6) in the strong sense, if u is the limit in the metric W_2^0 of a sequence of smooth vectors $\{u_k\}$ satisfying the boundary conditions (6) and such that $\mathcal{L}u_k$ tends to h in the metric W_2^0.

It is obvious that if a vector u satisfies the system (4) and the boundary conditions (6) in the strong sense, then it also satisfies them in the weak sense.

Theorem 1. *Let the domain Ω and the coefficients of the system (4) satisfy the conditions listed above; then for every prescribed vector h square-integrable in advance, the system (4) has a unique strong solution satisfying the boundary conditions (6) in the strong sense.*

Moreover, the strong solution u has a finite integral

$$\int_\Omega \left[u_x^2 + u_y^2\right]\, d\Omega$$

and satisfies the boundary conditions (6) in the mean.

Proof. We shall show that the range of values of the operator (4), defined on smooth vectors satisfying the boundary conditions (6), is dense in the space W_2^0. Suppose the contrary; then there exists a vector $v \in W_2^0$ such that

$$(v \cdot \mathcal{L}u) = 0 \tag{8}$$

or, if the Tricomi operator is denoted by T, then (8) can be rewritten as in the form

$$(v \cdot Tu) = (vK \cdot u). \tag{9}$$

Equality (9) is valid for all smooth vectors u satisfying the boundary condition (6). This is, in view of the formal self-adjointness of the operator T and the symmetry of the matrix K, the definition of a weak solution for the system $Tv = Kv$, satisfying conditions (6) in the weak sense. Using the classical results for the operator T and Lemma 2, we conclude that the weak solution v is equal to zero. Thus, the density of the range of the operator (4) in the space W_2^0 has been proved.

Now let h be some square-integrable vector; the vectors of the form $\mathcal{L}u$ are dense in W_2^0; therefore there exists a sequence of smooth vectors $\{u_k\}$, satisfying the boundary conditions (6), such that $\mathcal{L}u_k$ tends to h in the metric of W_2^0. It follows from Lemma 3 that the sequence of vectors $\{u_k\}$ converges in the metric of W_2^1. The limit of this sequence will be a strong solution. By the embedding theorem ([3]) we conclude that the strong solution of problem T for the system (4) satisfies the boundary conditions in the mean. Uniqueness is obtained by using Lemma 3.

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Received
29 XII 1961

CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.