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Abstract

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MATHEMATICS

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ON SOME SYSTEMS OF FUNCTIONS FORMING QUASI-POWER BASES IN SPACES OF ANALYTIC FUNCTIONS IN A DISC

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In the present paper we derive criteria for quasi-power bases (1) in the spaces $\mathfrak{A}(C_n)$ of analytic functions in the disc $C_r : \ |z| < r, \ 1 < r \le R$. For the systems under consideration such criteria were formulated by Yu. A. Kaz' min in (2) on the basis of the theory of infinite systems of linear equations. Using the representation of $\mathfrak{A}(C_r)$ in the form of a projective limit of Banach spaces $\mathfrak{B}_n = \mathfrak{B}(C_n)$ (3) and operators satisfying Fredholm theory, we obtain criteria that reveal a broader class of bases than those obtained in (2).

Consider the system of functions

$$f_k(z) = z^k + \sum_{n=0}^{\infty} a_{nk} z^n = (E + A) z^k \quad (k = 0, 1, ...), \tag{1}$$

where A is a linear operator defined on the basis elements:

$$A z^k = \sum_{n=0}^{\infty} a_{nk} z^n \quad (k = 0, 1, ...). \tag{2}$$

Let S be some subset of the set of all possible pairs of indices $\{(n, k)\}$ $(n, k = 0, 1, ...)$, and

$$q_{np} = \max_{k \ge p, \ (n,k) \notin S} a_{nk}. \tag{3}$$

Theorem 1. If the system (1) satisfies the conditions:

$$1)\ f_k(z) \in \mathfrak{A}(C_R)\ (k \ge 0), \quad 2)\ \varlimsup_{k\to\infty} \sum_{\substack{n=0\\(n,k)\in S}}^{\infty} |a_{nk}|\, r^{n-k} < 1, \quad 1 < r < R;$$

$$3)\ \lim_{p\to\infty} \varlimsup_{n\to\infty} \sqrt[n]{q_{np}} \le \frac{1}{R},$$

then it forms a quasi-power basis in each space $\mathfrak{A}(C_r)$ for $1 < r \le R$ if and only if it satisfies one of the conditions: a) $\{f_k(z)\}$ is complete in $\mathfrak{A}(C_r)$ for some r, $1 < r \le R$; b) $f_k(z)$ are ω-linearly independent over the coefficient space $\mathfrak{B}(C_r)$ for some r, $1 < r < R$.

Proof. Let r be arbitrary, $1 < r < R$. Choose δ so that $r < R - \delta < R$. From 3) it follows that there exists such a ρ_0 that

$$\varlimsup_{n\to\infty} \sqrt[n]{q_{n\rho_0}} < \frac{1}{R - \delta_0},$$

whence

$$|a_{nk}| \le \frac{C}{(R-\delta)^n} \quad (n = 0, 1, \ldots;\ (n,k) \notin S,\ k \ge \rho_0;\ C \text{ is constant}). \tag{4}$$

By virtue of 2), for some $\rho > 0$ and $k \geqslant k_1$,

$$\sum_{\substack{n=0\\(n,k)\in S}}^{\infty} |a_{nk}|\, r^{n-k} < 1 - 2\rho.$$

Choose k_0, $k_0 \geqslant \max(k_1, p_0)$, so that

$$C r^{-k_0} \sum_{n=0}^{\infty} [r(R-\delta)^{-1}]^n < \rho.$$

Represent the operator A in the form $A = A_1 + A_2$, where $A_2 = \|a_{nk}^{(2)}\|$, $a_{nk}^{(2)} = a_{nk}$ for $k \geqslant k_0$; $a_{nk} = 0$ for $k < k_0$. Then

$$\|A_2\|_{\mathfrak{B}(C_r)} = \sup_{k \geqslant k_0} \sum_{n=0}^{\infty} |a_{nk}|\, r^{n-k}$$

$$= \sup_{k \geqslant k_0} \left\{ \sum_{\substack{n=0\\(n,k)\in S}}^{\infty} |a_{nk}|\, r^{n-k} + \sum_{\substack{n=0\\(n,k)\notin S}}^{\infty} |a_{nk}|\, r^{n-k} \right\}$$

$$\leqslant 1-2\rho+\sup_{k\geqslant k_0}\sum_n \frac{C}{(R-\delta)^n}\frac{r^n}{r^k} \leqslant 1-2\rho+\frac{C}{r^{k_0}}\sum_{n=0}^{\infty}\left(\frac{r}{R-\delta}\right)^n \leqslant 1-\rho<1.$$

By virtue of 1), A_1 is a bounded finite-dimensional operator in $\mathfrak{B}(C_r)$. The operator $E+A=A_1+(E+A_2)$ is the sum of a finite-dimensional and a continuously invertible operator and therefore satisfies (^{4}) the Fredholm theory. Conditions a) and b) follow from the Fredholm theory and the mutual relation between the spaces $\mathfrak{B}(C_r)$ and $\mathfrak{A}(C_r)$ (^{3}).

Remark. In conditions a) and b) one may replace $f_k(z)$ by

$$f_k^*(z)=z^k-\sum_n a_{nk}z^n,$$

which means passing to the adjoint operator A'.

Theorem 2. *If the system*

$$f_k(z)=z^k+f^{(k)}z \qquad (k=0,1,\ldots), \tag{5}$$

where

$$f(z)=\sum_{n=0}^{\infty}\frac{a_n}{n!}z^n \tag{6}$$

satisfies the condition: for every $r>1$ *there exists a constant* $C(r)$ *such that*

$$|a_n|\leqslant C(r)r^n \qquad (n=0,1,\ldots), \tag{7}$$

then it forms a quasipower basis in each $\mathfrak{A}(C_r)$, $r>1$, *if and only if it satisfies one of conditions a), b) of Theorem 1.*

Proof. In this case we have $a_{nk}=a_{n+k}(n!)^{-1}$. Let S be the set of all pairs $\{(n,k)\}$ $(n,k=0,1,\ldots)$. Only condition 2) of Theorem 1 needs to be verified. Fixing r, $r>1$, and choosing ρ, $1<\rho<r$, we have

$$\sum_n |a_{nk}|\,r^{n-k}=\sum_n\frac{|a_{n+k}|}{n!}\,r^{n-k}\leqslant\sum_{n=0}^{\infty}\frac{C(\rho)\rho^{n+k}}{n!}\,r^{n-k}=$$

$$=C(\rho)\left(\frac{\rho}{r}\right)^k\sum_{n=0}^{\infty}\frac{(r\rho)^k}{n!}=C(\rho)\left(\frac{\rho}{r}\right)^k e^{r\rho}\to 0 \qquad (k\to\infty).$$

The operator A is completely continuous in each space $\mathfrak{B}(C_r)$, $r>1$.

Remark. Condition (7) is satisfied, for example, by the sequences $a_n=C_n^s$, where s is any positive number, and $a_n=\varphi(n)$, where $\varphi(z)$ is an entire function of exponential order of growth and of zero type. Indeed, in the second case we have $|\varphi(n)|\leqslant C(\alpha)e^{n\alpha}$ $(\alpha>0,\ n\geqslant 0)$. Choosing $\alpha=\log r$, $r>1$, we arrive at inequality (7).

Example. Let s be an arbitrary natural number. We shall show that the system of functions (5), where

$$f(z) = z^s e^z, \tag{8}$$

is a quasipower basis in each space $\mathfrak{A}(C_r)$, $r > 1$.

In this case $a_n = n(n+1)\cdots(n-s+1)$ $(n = s, s+1, \dots)$ and $a_n = 0$ for $n < s$. Condition (7) is fulfilled; therefore it is enough to verify that in some space $\mathfrak{B}(C_r)$, $r > 1$, the value -1 is not an eigenvalue of the operator A. Suppose that $\varphi(z) = \sum_k b_k z^k$ satisfies the equation

$$A\varphi(z) = A\sum_k b_k z^k = \sum_k b_k (z^s e^z)^{(k)} = -\varphi(z).$$

Applying Leibniz' formula and summing, we obtain

$$-\varphi(z) = e^z\big[C_s^0\varphi(1)z^s + C_s^2\varphi'(1)z^{s-1} + \dots + C_s^s\varphi^{(s)}(1)z^{s-s}\big]. \tag{9}$$

Differentiating k times $(k = 0, 1, \dots, s)$ and replacing z by 1, we obtain a homogeneous linear system of $s+1$ equations with unknowns $\varphi(1), \varphi'(1), \dots, \varphi^{(s)}(1)$. The determinant of this system has the form $|\beta_{ij}e + \delta_{ij}|$, where β_{ij} are natural numbers and δ_{ij} is the Kronecker symbol. On replacing e by x, it is not identically equal to zero and therefore, taking into account the transcendence of the number e, is nonzero. The system has only the trivial solution $\varphi^{(k)}(1) = 0$ $(k = 0, 1, \dots, s)$. On the basis of representation (9) we conclude that $\varphi(z) \equiv 0$.

Theorem 3. *The system of functions*

$$f_k(z) = z^k f^{(k)}(\zeta_k z) \qquad (k = 0, 1, \dots), \tag{10}$$

where

$$f(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n,$$

forms a quasi-power basis in every $\mathfrak{A}(C_r)$, $0 < r \le R$, *if: 1)* $a_n \ne 0$, $n = 0, 1, \dots$; *2) there exist a sequence* $\{\beta_n\}_0^\infty$, $\lim_{n\to\infty} |\beta_n| = \infty$, $\lim_{n\to\infty} |\beta_n|\, 0$, *and a number* θ, $0 < \theta \le 1/2$, *such that*

$$\varlimsup_{k\to\infty} |\beta_k a_k - 1| = 1 - 2\theta, \qquad \varlimsup_{k\to\infty} |\zeta_k| < \frac{1}{R}\ln\frac{2}{2-\theta}. \tag{11}$$

Proof. We have

$$a_{nk} = 0,\ n < k; \qquad a_{nk} = a_k - 1,\ n = k; \qquad a_{nk} = \frac{a_n}{(n-k)!}\zeta_k^{n-k}, \quad n > k.$$

Let first $\beta_n = 1$, $n = 0, 1, \dots$. There exist $k_0 > 0$ and $0 < \delta < \dfrac{2}{2-\theta}$ such that for $k \ge k_0$ we have $|a_k| \le 2 - \theta$, $|a_k - 1| \le 1 - \theta$, and

$$|\zeta_k| \le \frac{1}{R}\ln\left(\frac{2}{2-\theta} - \delta\right).$$

Then, for $k \geq k_0$ and $r < R$,

$$\sum_n |a_{nk}| r^{n-k} = |a_k - 1| + \sum_{n=1}^{\infty} \frac{|a_{n+k}|}{n!} |\zeta_k|^n r^n \leq$$

$$\leq 1 - \theta + (2-\theta) \sum_{n=1}^{\infty} \frac{1}{n!} \left[\frac{1}{R} \ln \left(\frac{2}{2-\theta} - \delta \right) \right]^n r^n \leq$$

$$\leq 1 - \theta + (2-\theta) \left[\frac{2}{2-\theta} - \delta - 1 \right] = 1 - (2-\theta)\delta < 1.$$

By virtue of the triangular form of the system and condition 1), the system $\{f_k(z)\}$ is ω-linearly independent. The passage from a_k to $\beta_k a_k$ is carried out by the continuously invertible operator $Bz^k = \beta_k z^k$ $(k = 0, 1, \ldots)$.

Theorem 4. *Let*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathfrak{A}(C_R), \qquad R > 1.$$

In order that the system

$$f_k(z) = z^k + \int_0^z \int_0^z \cdots \int_0^z f(z)\,(dz)^k \qquad (k = 0, 1, \ldots) \tag{12}$$

form a quasi-power basis in every $\mathfrak{A}(C_r)$, $1 < r \leq R$, it is necessary and sufficient that $a_0 \neq -k$ $(k = 0, 1, \ldots)$.

The sufficiency of this condition is indicated in (^{2}) (Theorem 5). In the present case

$$\sum_n |a_{nk}|\, r^{n-k} \leq \frac{1}{k!} f_0(r) \to 0, \quad \text{where } f_0(z) = \sum_{n=0}^{\infty} |a_n| z^n.$$

The operator A is completely continuous in $\mathfrak{B}(C_n)$. If $a_0 = -k!$, then the adjoint operator $E + A'$ has a nontrivial zero of the form $\sum_{i=0}^{k} \alpha_i \dfrac{1}{\xi^{i+1}}$, and the system is not a basis.

Theorem 5. Let $f(z) = \sum_n a_n z^n \in \mathfrak{A}(C_r)$. If $\alpha = \max_k |\zeta_k|$, $\alpha < R$, then the system

$$f_k(z) = z^k + \zeta_k f(\zeta_k z) \qquad (k = 0, 1, \ldots) \tag{13}$$

is a quasi-power basis in every $\mathfrak{A}(C_r)$, $1 < r \leq \dfrac{R}{\alpha}$, provided one of the conditions a), b) of Theorem 1 is satisfied.

Proof. In this case

$$\sum_n |a_{nk}|\, r^{n-k} = \sum_n |a_n|\, |\zeta_k|^{n+1} r^{n-k} \le \alpha \frac{1}{r^k} f_0(\alpha r) \to 0 \quad (1 < r < \frac{R}{\alpha}).$$

Theorem 6. The system

$$f_k(z) = z^k + \sum_{i=1}^{s} \varepsilon_i^{(k)} \varphi_i(z) \tag{14}$$

is a quasi-power basis in every $\mathfrak{A}(C_r)$ for $1 < r \le R$, if:

1) $\varphi_i(z) = \sum_{k=0}^{\infty} a_k^{(i)} z^k \in \mathfrak{A}(C_R)$ $(i = 0, 1, \ldots)$,

2) $\sup\limits_k \sum\limits_m \left| \sum\limits_{i=1}^{s} \varepsilon_i^{(k)} a_m^{(i)} \right| r^{m-k} < \infty$ for all r, $1 < r < R$;

3) $\det \left| \sum\limits_k a_k^{(i)} \varepsilon_j^{(k)} + \delta_{ij}^s \right| \ne 0.$

The operator A in this case is finite-dimensional and bounded in $\mathfrak{B}(C_r)$, $1 < r < R$, and by condition 3) -1 is not its eigenvalue.

In conclusion, let us note that in some cases it is impossible to formulate necessary and sufficient conditions for quasi-power bases in the spaces $\mathfrak{A}(C_r)$, $1 < r \le R$, in terms only of conditions on the absolute values of the elements of the matrix $\|a_{ik}\|$. What has been said applies, for example, to a system of the form $f_k(z) = z^k f(\zeta_k z)$. Indeed, the system $\{z^k e^{az}\}_{k=0}^{\infty}$ is a quasi-power basis in every space $\mathfrak{A}(C_r)$ for $r > 0$, whereas the system $\{z^k e^{(-1)^k az}\}_{k=0}^{\infty}$ is a quasi-power basis in the space $\mathfrak{A}(C_r)$ for $r \le \pi/4a$ and is not a basis in $\mathfrak{A}(C_r)$ for $r > \pi/4a$.

Let us also note that in cases where conditions a), b) of Theorem 1 are not satisfied, for all the systems considered

$$f_k(z) = z^k + \sum_{n=0}^{\infty} a_{nk} z^n$$

one can assert that the systems

$$z^k + \lambda \sum_{n=0}^{\infty} a_{nk} z^n, \quad \lambda \text{ complex},$$

form quasi-power bases in $\mathfrak{A}(C_r)$ for all λ except for a discrete set.

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Note: Figure translations are in progress. See original paper for figures.