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Abstract

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ON HOMOGENEOUS DIFFERENTIAL-DIFFERENCE SCHEMES OF SECOND-ORDER ACCURACY FOR PARABOLIC AND HYPERBOLIC EQUATIONS WITH DISCONTINUOUS COEFFICIENTS

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1. In this note we shall consider the solution, by the method of lines, of boundary value problems for the equations

$$L'(u) \equiv \rho(x,t)u_t - (k(x,t)u_x)_x - \chi(x,t)u_x + q(x,t)u - f(x,t) = 0, \quad 0 < x < l, \qquad 0 < t < T; \tag{1,1'}$$

$$L''(u) \equiv \rho(x,t)u_{tt} - (k(x,t)u_x)_x - \chi(x,t)u_x + q(x,t)u - f(x,t) = 0. \quad 0 < x < l, \qquad 0 < t < T, \tag{1,1''}$$

with boundary conditions of the form

$$u(0,t) = \varphi_1(t), \qquad u(l,t) = \varphi_2(t), \qquad 0 \le t \le T, \tag{1,2a}$$

or of the form

$$k(0,t)u_x(0,t) = \psi_1(t), \qquad u(l,t) = \psi_2(t), \qquad 0 \le t \le T, \tag{1,2b}$$

and initial conditions

$$u(x,0) = \omega(x), \qquad 0 \le x \le l; \tag{1,3'}$$

$$u(x,0) = \omega_1(x), \qquad u_t(x,0) = \omega_2(x), \qquad 0 \le x \le l, \tag{1,3''}$$

respectively. Here $\rho(x,t) \geq \rho_{\min} > 0$, $k(x,t) \geq k_{\min} > 0$, $q(x,t) \geq 0$. It is assumed that ρ, k, q, χ, f, together with their derivatives, undergo jump discontinuities on a finite number of straight lines $x = X_k$, $0 < X_1 < X_2 < \cdots < X_N < l$. We shall also write the equations of the boundary lines in the form $x \equiv X_0 = 0$, $x \equiv X_{N+1} = 0$. The functions $\rho, k, \chi, q, f, \varphi_1, \varphi_2, \psi_1, \psi_2, \omega, \omega_1, \omega_2$ will be assumed such that each of the boundary value problems $((1,1'), (1,2a,b), (1,3'))$; $((1,1''), (1,2a,b), (1,3''))$ has in the rectangle $\Pi:\ 0 \leq x \leq l,\ 0 \leq t \leq T$ a continuous unique solution satisfying on the straight lines $x \equiv X_k$, $k = 1, 2, \dots, N$, the conjugacy conditions

$$u\big|_{X_k-0} = u\big|_{X_k+0}, \qquad (ku_x)\big|_{X_k-0} = (ku_x)\big|_{X_k+0} \tag{1,4}$$

and possessing uniformly bounded derivatives $u_x, u_{xx}, u_{xxx}, u_{xxxx}, u_t, u_{tx}$ between the straight lines $x \equiv X_k$, $k = 0, 1, \dots, N+1$*. In the case of boundary conditions (1,2b) we shall regard the prescribed functions and the solutions of the boundary value problems as smoothly continued through the straight line $x \equiv X_0 = 0$ to the left, with preservation of the differential properties. In the estimates we shall also use the uniform boundedness of certain derivatives of the functions $\rho, k, \chi, q, f, \omega, \omega_1, \omega_2$ between the straight lines $x \equiv X_k$, $k = 0, 1, \dots, N+1$, and the uniform boundedness of certain derivatives of the functions $\varphi_1, \varphi_2, \psi_1, \psi_2$.

2. In the case of boundary conditions (1,2a) we take the mesh of nodes $S_1\{x_i = ih,\ i = 0, 1, \dots, (n+1);\ (n+1)h = l\}$. In the case of boundary conditions (1,2b) we take the mesh of nodes $S_2\{x_i = (i - \frac{1}{2})h,\ i = 0, 1, \dots, (n+1);\ (n+1)h = l\}$. We shall assume that each of the straight lines $x \equiv X_k$, $k = 1, \dots, N+1$, passes through one of the mesh nodes**.

* The existence and uniqueness of the solution and the investigation of differential properties for an equation with discontinuous coefficients were considered in ([2,5]).

** In order that this can be achieved with a step h common to the entire interval $0 \leq x \leq l$, it may be necessary to make a change of the variable x.

We shall approximate equations $(1,1')$ and $(1,1'')$ by the equations

$$L'_h(u_i) = \tilde{\rho}_i \frac{du_i}{dt} - \frac{1}{h^2}\left[k_{i+1/2}(u_{i+1} - u_i) - k_{i-1/2}(u_i - u_{i-1})\right] - -\varkappa_{i+1/2}\frac{u_{i+1} - u_i}{2h} - \varkappa_{i-1/2}\frac{u_i - u_{i-1}}{2h} + \tilde{q}_i u_i - \tilde{f}_i = 0; \tag{2,1'}$$

$$L''_h(u_i) = \tilde{\rho}_i \frac{d^2u_i}{dt^2} - \frac{1}{h^2}\left[k_{i+1/2}(u_{i+1} - u_i) - k_{i-1/2}(u_i - u_{i-1})\right] - -\varkappa_{i+1/2}\frac{u_{i+1} - u_i}{2h} - \varkappa_{i-1/2}\frac{u_i - u_{i-1}}{2h} + \tilde{q}_i u_i - \tilde{f}_i = 0, \tag{2,1''}$$

$$\tilde{\rho}_i = \frac{1}{2}[\rho(x_i - 0, t) + \rho(x_i + 0, t)], \quad \tilde{q}_i = \frac{1}{2}[q(x_i - 0, t) + q(x_i + 0, t)],$$

$$\tilde{f}_i = \frac{1}{2}[f(x_i - 0, t) + f(x_i + 0, t)], \quad k_{i\pm1/2} = k(x_i \pm h/2, t), \quad \varkappa(x_i \pm h/2, t) = \varkappa_{i\pm1/2}, \quad 1 \le i \le n.$$

The boundary conditions $(1, 2a)$ and $(1, 2)$ shall be replaced, respectively, by the conditions

$$u_0(t) = \varphi_1(t), \qquad u_{n+1}(t) = \varphi_2(t), \qquad 0 \le t \le T; \tag{2,2a}$$

$$l_h(u_i) \equiv k_{1/2}\frac{u_1 - u_0}{h} - \varphi_1(t) = 0, \qquad u_{n+1} = \varphi_2(t), \qquad 0 \le t \le T. \tag{2,2 }$$

The initial conditions $(1, 3')$ and $(1, 3'')$ shall be replaced, respectively, by the conditions

$$u_i(0) = \omega(x_i), \qquad i = 0, 1, \dots, n + 1; \tag{2,3'}$$

$$u_i(0) = \omega_1(x_i), \qquad u_i'(0) = \omega_2(x_i), \qquad i = 0, 1, \dots, n + 1. \tag{2,3''}$$

3. Let $u^{(a)}(x, t)$ and $u^{()}(x, t)$ denote the solutions of the boundary-value problems with boundary conditions $(1, 2a)$ and $(1, 2)$, and let $u_i^{(a)}(t)$ and $u_i^{()}(t)$ be the solutions of the corresponding differential-difference boundary-value problems with boundary conditions $(2, 2a)$ and $(2, 2)$*.

Theorem. Under the formulated conditions, the errors $\delta_i^{(a)} = u^{(a)}(x_i, t) - u_i^{(a)}(t)$, $\delta_i^{()} = u^{()}(x_i, t) - u_i^{()}(t)$ are of order $O(h^2)$ as $h \to 0$.

Proof. We give the outline of the proof. First we determine the order of approximation for the equations and boundary conditions, i.e., the order with respect to h of the residual terms obtained when the exact solutions of the differential boundary-value problems are substituted into the differential-difference equations and boundary conditions. Then we obtain differential-difference boundary-value problems for the errors $\delta_i^{(a)}$ and $\delta_i^{()}$, after which we estimate $\delta_i^{(a)}$ and $\delta_i^{()}$, using the "energy integral" that we previously applied in the case of equations with continuous coefficients.

4. Let us find the approximation error for equations $(2, 1')$, $(2, 1'')$. If $x = x_i$ is not a point of discontinuity, then, applying Taylor expansions for $k(x, t)$, $\varkappa(x, t)$, and $u(x, t)$, we easily obtain

$$L'_h(u(x_i,t)) = L'(u(x,t))_{x=x_i} + O(h^2) = O(h^2); \qquad (4, 1'_1)$$

$$L''_h(u(x_i,t)) = L''(u(x,t))_{x=x_i} + O(h^2) = O(h^2). \qquad (4, 1''_1)$$

If $x = x_i$ lies on one of the internal straight lines $x = X_k$, i.e., is a point of discontinuity, then, integrating $(1, 1')$ with respect to x from $x_i - h/2$ to x_i and from x_i to $x_i + h/2$, we obtain:

$$\int_{x_i-h/2}^{x_i} \rho(x,t)\frac{\partial u}{\partial t}\,dx = \left[k(x,t)\frac{\partial u}{\partial x}\right]_{x_i-0} - \left[k(x,t)\frac{\partial u}{\partial x}\right]_{x_i-h/2} +$$

* The actual solution of these problems for the systems of ordinary differential equations $(2, 1')$ and $(2, 1'')$ can be carried out by numerical integration, as in the case of the more general method of integral relations (1), using any standard program.

$$+\int_{x_i-h/2}^{x_i} \left[\varkappa(x,t)\frac{\partial u}{\partial x} - q(x,t)u + f(x,t)\right]\,dx,$$

$$\int_{x_i}^{x_i+h/2} \rho(x,t)\frac{\partial u}{\partial t}\,dx = \left[k(x,t)\frac{\partial u}{\partial x}\right]_{x_i+h/2} - \left[k(x,t)\frac{\partial u}{\partial x}\right]_{x_i+0} + \int_{x_i}^{x_i+h/2} \left[\varkappa(x,t)\frac{\partial u}{\partial x} - q(x,t)u + f(x,t)\right]\,dx.$$

We transform these equations, using rectangular quadrature formulas of the form

$$\int_{x_i}^{x_i+h/2} F(x)\,dx = \frac{h}{2}F(x_i+0) + \frac{h^2}{2}F'(\xi_1),$$

$$\int_{x_i-h/2}^{x_i} F(x)\,dx = \frac{h}{2}F(x_i-0) + \frac{h^2}{2}F'(\xi_2)*$$

and the formulas

$$\left.\frac{\partial u}{\partial x}\right|_{x_i-h/2} = \frac{u(x_i,t) - u(x_i-h,t)}{h} + O(h^2),$$

$$\left.\frac{\partial u}{\partial x}\right|_{x_i+h/2} = \frac{u(x_i+h,t) - u(x_i,t)}{h} + O(h^2),$$

and then subtract one from the other and, using conditions (1,4), divide the result by h. This gives

$$L'_h(u(x_l,t)) = O(h). \qquad (4, 1'_2)$$

Similarly we obtain:

$$L''_h(u(x_l,t)) = O(h). \qquad (4, 1''_2)$$

For the first boundary condition (2,26) we obtain

$$l_h(u(x_l,t)) \equiv k_{1/2}\frac{u(x_1,t)-u(x_0,t)}{h} - \psi_1(t) = O(h^2). \qquad (4{,}26)$$

The remaining boundary and initial conditions are approximated exactly.

5. Now for $\delta_i^{(a)}$ and $\delta_i^{(b)}$ we obtain differential-difference boundary-value problems and estimate $\delta_i^{(a)}$ and $\delta_i^{(b)}$ by means of the "energy integral." Let us consider in detail, for example, the case of the first boundary-value problem for equation $(1, 1'')$. We have

$$\tilde{\rho}_i\frac{d^2\delta_i}{dt^2} = \frac{1}{h^2}\left[k_{i+1/2}(\delta_{i+1}-\delta_i) - k_{i-1/2}(\delta_i-\delta_{i-1})\right] +$$

$$+\varkappa_{i+1/2}\frac{\delta_{i+1}-\delta_i}{2h} + \varkappa_{i-1/2}\frac{\delta_i-\delta_{i-1}}{2h} - \tilde{q}_i\delta_i + R_i h^{\sigma_i}, \qquad 0 < i < n+1; \quad (5{,}1")$$

$$\delta_0(t) = 0, \qquad \delta_{n+2}(t) = 0; \qquad (5{,}2a)$$

$$\delta_i(0) = 0, \qquad \delta'_i(0) = 0, \qquad 0 \le i \le n+1, \qquad (5{,}3)$$

where $\sigma_i = 2$ if x_i is a point of continuity, and $\sigma_i = 1$ if x_i is a point of discontinuity, while $R_i = O(1)$. Consider the "energy integral"

$$I(t) = \sum_{i=1}^{n}\tilde{\rho}_i\dot{\delta}_i^2 + \sum_{i=0}^{n}k_{i+1/2}\left(\frac{\delta_{i+1}-\delta_i}{h}\right)^2 + \sum_{i=1}^{n}\tilde{q}_i\delta_i^2. \qquad (5{,}4)$$

* In the right-hand sides of these equalities, when integrating $\varkappa(x,t)\partial u/\partial x$, instead of $x_i \pm 0$ one should take $x_i \pm h/2$, respectively.

Differentiating, we obtain, using equations $(5, 1')$ and the boundary conditions $(5, 2a)$:

$$\frac{dI}{dt} = 2\sum_{i=1}^{n}\left(\chi_{i+1/2}\frac{\delta_{i+1}-\delta_i}{2h} + \chi_{i-1/2}\frac{\delta_i-\delta_{i-1}}{2h} + R_i h\delta_i\right)\delta_i' +$$

$$+\sum_{i=1}^{n}\tilde{\rho}_i\delta_{i'}^2 + \sum_{i=0}^{n}k_{i+1/2}'\left(\frac{\delta_{i+1}-\delta_i}{h}\right)^2 + \sum_{i=1}^{n}\tilde{q}_i'\delta_i^2. \qquad (5,5)$$

Without loss of generality, in what follows we shall assume that there is only one line of discontinuity, $x = x_{i_0}$. Accordingly, from the first sum on the right-hand side of $(5,5)$, in making estimates we shall single out the term with the first power of h, i.e. $R_{i_0}\delta_{i_0}'h$. Using the relation (see (3))

$$|\delta_i| \le \sqrt{\frac{x_i(l-x_i)}{lk_{\min}}}\sqrt{I(t)}\,h^{1/2} \qquad (5,6)$$

and expression $(5,4)$, we obtain

$$\sum_{i=1}^{n}\tilde{\rho}_i\delta_{i'}^2 + \sum_{i=0}^{n}k_{i+1/2}'\left(\frac{\delta_{i+1}-\delta_i}{h}\right)^2 + \sum_{i=1}^{n}\tilde{q}_i'\delta_i^2 \le C_1 I(t), \qquad C_1 = \text{const} > 0.$$

Using the Cauchy inequality, we find

$$\left|2\sum_{\substack{i=1\\ i\ne i_0}}^{n}R_i\delta_i'h^2\right| + 2R_{i_0}\delta_{i_0}'h \le C_2 h^{3/2}\sqrt{I(t)} + 2R_{i_0}\delta_{i_0}'h,$$

$$\left|2\sum_{i=1}^{n}\left(\chi_{i+1/2}\frac{\delta_{i+1}-\delta_i}{2h} + \chi_{i-1/2}\frac{\delta_i-\delta_{i-1}}{2h}\right)\delta_i'\right| \le C_3 I(t).$$

As a result we obtain

$$\frac{dI}{dt} \le C_4 I + C_4 h^{3/2}\sqrt{I} + 2hR_{i_0}\delta_{i_0}'.$$

Integrating this inequality, taking into account the initial conditions $I(0) = 0$, $\delta_{i_0}(0) = 0$, and applying inequality $(5,6)$, we obtain

$$I - C_4\int_0^t I\,dt \le C_6 h^{3/2}\left(\int_0^t \sqrt{I}\,dt + \sqrt{I}\right). \qquad (5,7)$$

Multiplying the left-hand side of this inequality by $e^{-C_4 t}$, we obtain

$$\frac{d}{dt}\left(e^{-C_4 t}\int_0^t I\,dt\right) \leq h^{3/2} C_6 \left(\int_0^t \sqrt{I}\,dt + \sqrt{I}\right).$$

Integrating this inequality from 0 to t, substituting the result in $(5, 7)$, and then applying the Cauchy inequality, we find

$$\sqrt{\max_{0\leq t\leq T} I(t)} \leq C h^{3/2}.$$

Consequently, by virtue of $(5, 6)$, it follows that

$$|\delta_i| \leq h^2 C^* \sqrt{\frac{x_i(l - x_i)}{l}}.$$

Other boundary-value problems are considered analogously; moreover, the energy integral for equation $(1, 1')$ is written in the form of expression (10) on p. 11 in ([3]).

Homogeneous difference schemes for equations of parabolic type with discontinuous coefficients were considered by another method in ([4]).

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Note: Figure translations are in progress. See original paper for figures.