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# MATHEMATICS

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**Abstract**

**Full Text**

**MATHEMATICS**

**V. G. MAZ' YA**

**ON THE NEGATIVE SPECTRUM OF A  
MULTIDIMENSIONAL SCHRÖDINGER OP-  
ERATOR**

*(Presented by Academician V. I. Smirnov, January 20, 1962)*

In the present note we formulate necessary and sufficient conditions for the discreteness, finiteness, and infiniteness, for all  $h > 0$ , of the negative spectrum of the operator

$$M_h u = -h\Delta u - P(x)u \quad (P(x) \geq 0)$$

in  $L_2(R_n)$  ( $n \geq 3$ ).<sup>\*</sup> Here  $R_n$  is  $n$ -dimensional Euclidean space.

Let  $\Omega$  be an open subset of  $R_n$ . By  $\overset{\circ}{C}^{(1)}(\Omega)$  we denote the set of functions continuously differentiable in  $\Omega$  and equal to zero outside some ball and near the boundary of  $\Omega$ . By the space  $\overset{\circ}{L}_2^{(1)}(\Omega)$  ( $\overset{\circ}{W}_2^{(1)}(\Omega)$ ) we shall mean the closure of  $\overset{\circ}{C}^{(1)}(\Omega)$  in the metric

$$\int_{\Omega} (\text{grad } u)^2 dx \quad \left( \int_{\Omega} (\text{grad } u)^2 dx + \int_{\Omega} u^2 dx \right).$$

We also introduce the space  $L_2(P(x); \Omega)$  of functions for which

$$\|u\|_{L_2(P(x); \Omega)}^2 = \int_{\Omega} P(x)u^2 dx < \infty.$$

Let  $E$  be an arbitrary bounded closed subset of  $\Omega$  with boundary  $\Gamma E$ . By  $c(\Omega \setminus E)$  we shall mean the capacity of the condenser  $\Omega \setminus E$ , i.e. the number

$$\inf \frac{1}{(n-2)\omega_n} \int_{\Omega \setminus E} (\text{grad } u)^2 dx,$$

where  $\omega_n$  is the surface area of the unit  $(n-1)$ -dimensional sphere and the infimum is taken over all functions  $u(x) \in C(\Omega \setminus E) \cap C^{(1)}(\Omega \setminus E)$  equal to one

on  $\Gamma E$  and to zero near  $\Gamma\Omega$  and outside some ball. By  $\text{cap } E$  we denote the capacity of the set  $E$ , i.e.  $c(R_n \setminus E)$ .

In Theorem 1 conditions are given for the positivity of the operator  $M_1$ .

**Theorem 1.** *For the validity of the inequality*

$$\int_{\Omega} P(x)u^2 dx \leq \int_{\Omega} (\text{grad } u)^2 dx, \quad (1)$$

where  $u(x) \in \mathring{L}_2^{(1)}(\Omega)$ , it is sufficient that for every bounded closed set  $E \subseteq \Omega$  the inequality

$$\int_E P(x) dx \leq \frac{n-2}{4} \omega_n c(\Omega \setminus E). \quad (2)$$

hold. The inequality

$$\int_E P(x) dx \leq (n-2)\omega_n c(\Omega \setminus E) \quad (3)$$

is necessary for the validity of (1).

\* An exhaustive bibliography on the questions considered can be found in <sup>(1)</sup>.

**Corollary.** For Friedrichs' inequality

$$\int_{\Omega} u^2 dx \leq \text{const} \int_{\Omega} (\text{grad } u)^2 dx \quad \left(u \in \mathring{L}_2^{(1)}(\Omega)\right) \quad (4)$$

to hold, it is necessary and sufficient that the condition

$$\sup_{E \subseteq \Omega} \frac{\text{mes}_n E}{\text{cap}(\Omega \setminus E)} < \infty \quad (5)$$

be satisfied.

In what follows we shall assume that the function  $P(x)$  is bounded in every ball of the space  $R_n$ .

**Theorem 2.** For complete continuity of the embedding operator  $\mathring{W}_2^{(1)}(R_n)$  into  $L_2(P(x); R_n)$ , it is necessary and sufficient that the condition

$$\lim_{R \rightarrow \infty} \sup_{\substack{E \subset CS_R, \\ d(E) \leq 1}} \frac{\int_E P(x) dx}{\text{cap } E} = 0, \quad (6)$$

be satisfied, where the sup is taken over closed sets  $E$  of diameter  $d(E) \leq 1$ , situated outside the ball  $S_R$  of radius  $R$  with center at the origin.

**Theorem 3.** For complete continuity of the embedding operator  $\mathring{L}_2^{(1)}(R_n)$  into  $L_2(P(x); R_n)$ , it is necessary and sufficient that the condition

$$\lim_{R \rightarrow \infty} \sup_{E \subset CS_R} \frac{\int_E P(x) dx}{\text{cap } E} = 0, \quad (7)$$

be satisfied, where the sup is taken over bounded closed sets  $E$ , situated outside the ball  $S_R$  of radius  $R$  with center at the origin.

From the general theorem of M. Sh. Birman <sup>(1)</sup> it follows that Theorems 2 and 3 admit the following equivalent formulations.

**Theorem 2'.** In order that the negative spectrum of the operator  $M_h$  be discrete for all  $h > 0$ , it is necessary and sufficient that condition (6) be satisfied.

**Theorem 3'.** In order that the negative spectrum of the operator  $M_h$  be finite for all  $h > 0$ , it is necessary and sufficient that condition (7) be satisfied.

The following theorem gives a criterion for infinitude of the negative spectrum of the operator  $M_h$  for all  $h > 0$ .

**Theorem 4.** The condition

$$\sup_{E \subset R_n} \frac{\int_E P(x) dx}{\text{cap } E} = \infty \quad (8)$$

is necessary and sufficient for the infinitude of the negative spectrum of the operator  $M_h$  for all  $h > 0$ .

We note that some sufficient conditions for discreteness, finiteness, and infinitude of the negative spectrum of the operator  $M_h$  for all  $h > 0$  were given by M. Sh. Birman <sup>(1)</sup>.

Related in subject to the present note is the paper of A. M. Molchanov <sup>(2)</sup>, in which a necessary and sufficient condition was found for discreteness of the spectrum of the Schrödinger operator  $-\Delta u + Q(x)u$  with potential  $Q(x)$  bounded below.

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## REFERENCES

<sup>1</sup> M. Sh. Birman, *Mat. sborn.*, **55**, No. 22 (1961).

<sup>2</sup> A. M. Molchanov, *Tr. Mosk. matem. obshch.*, **2**, 169 (1953).

*Note: Figure translations are in progress. See original paper for figures.*

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