# MATHEMATICS

1962

SovietRxiv

## Abstract

## Full Text

## MATHEMATICS

### S. N. SLUGIN

# MONOTONE PROCESSES OF TWO-SIDED APPROXIMATIONS IN A PARTIALLY ORDERED CONVERGENCE GROUP

*(Presented by Academician S. L. Sobolev on 28 V 1962)*

The indicated methods of approximate solution have been investigated chiefly for equations posed on a conditionally $\sigma$-complete linear structure ($^1$), and also in partially ordered topological groups ($^2$). In the present work objects satisfying more general requirements are introduced. A theory of these methods is constructed for the case when the equation is posed on a partially ordered set ($^1$) or on a structure, in general different from a conditionally $\sigma$-complete one, and here the requirement expressed by relation (4) is essential.

1. Let, in the additively written group $X$, a class of essentially positive elements $x > 0$ be distinguished. We shall call the commutative group $X$ partially ordered (cf. ($^1$)) if from the relation $x > 0$ it follows that $x \neq 0$, and from $x > 0$, $y > 0$ it follows that $x + y > 0$. An example of such a group, different from a structure, is the linear set of differentiable functions with the natural sense of comparability of the functions themselves.

We put $|x| = x$, $|y| = -y$, if $x \geq 0 > y$; $x > y$ if $x - y > 0$, etc.; $[x, y]$ denotes the segment ($^1$). $[x, y]_K = (x, y) \cap K$, where $x, y \in K \subset X$.

Let there exist in $X$ classes of countable sequences converging to zero, $x_n \to 0$ as $n \to \infty$, and converging in themselves, $x_m - x_n \to 0$ as $m > n \to \infty$. We put $x_n \to x$ if $x_n - x \to 0$.

**Definition 1.** A partially ordered group $X$ satisfying the conditions: a) if $x_n = x$, $x_n \to 0$, then $x = 0$; b) if $0 \leq x_n \leq y_n \to 0$, then $x_n \to 0$; c) if $x_n \to 0$, $y_n \to 0$, then $x_n - y_n \to 0$; d) if $0 \leq x_m - x_n \leq y_n \to 0$ as $m > n \to \infty$, then $x_n$ converges in itself; e) convergence in $X$ and convergence in itself are equivalent; f) if $x_n \geq 0$, $x_n \to x$, then $x \geq 0$, will be called a **partially ordered convergence group** (p.o.c. group).

**Corollary 1.** *In a p.o.c. group a stationary sequence $x_n = x$ converges to $x$. Let $x_n \to x$, $y_n \to y$. Then: $x_n \pm y_n \to x \pm y$; if $x_n \geq y_n$, then $x \geq y$; if $0 \leq z \leq x_n$, $x = 0$, then $z = 0$; if $x_n \leq z_n \leq y_n$; $x = y$, then $z_n \to x$; if $x_n \geq x_{n+1}$ ($\leq$), then $x_n \geq x$ ($\leq$). The limit is unique. The principle of nested* *segments holds: if* $u_n \leq u_{n+1} \leq v_{n+1} \leq v_n$, $v_n - u_n \to 0$, *then there exists a unique element* $u \in [u_n, v_n]$ *for all* $n$, *and* $u_n \to u$, $v_n \to u$.

2. We indicate a special case of a p.o.c. group possessing a semi-ordering. Let in a $K_0$-group ($^2$) (semi-ordered; a structure, in general incomplete) there exist classes of countable sequences converging in themselves and positive sequences converging to zero. We require:

c′) If $x_n \to 0$, $y_n \to 0$, then $x_n + y_n \to 0$.

We put $x_n \to x$ if $|x_n - x| \to 0$.

**Definition 2.** If in a $K_0$-group $X$ conditions a), b), c′) are fulfilled for positive sequences and conditions d), e) for arbitrary sequences, then we shall call $X$ a **semi-ordered convergence group** (s.o.c. group).

**Corollary 2.** In a p. c. group, from the relation $\dot{x}_n \to \dot{x}$ it follows that $|x_n| \to |x|$; every monotone convergent sequence is (0)-convergent ($^1$) to the same limit. A p. c. group is a c. u. c. group.

A special case of a p. c. group is a countably complete $KT$-group ($^2$) with the first axiom of separability and, in particular, a $K_0$-group that is a countably complete metric space with a metric invariant with respect to shift and increasing: if $|x| \leq |y|$, then $\rho(x, 0) \leq \rho(y, 0)$. A $K_0$-group with (0)-convergence, countably complete with respect to this convergence, also belongs to a p. c. group; in particular, $K_\sigma$- and $K$-spaces ($^1$).

3. Let us single out one class of linear spaces belonging to a p. c. group.

**Definition 3.** If a $K$-lineal ($^1$) $X$ is a $(bk)$-complete space, lattice-normed ($^1$) by means of a $K$-lineal $Z$ in such a way that from the relation $|x| \leq |y|$ in $X$ it follows that $\|x\| \leq \|y\|$ in $Z$ for generalized norms, then $X$ will be called a $K_{bk}$-lineal.

By convergence in a $K_{bk}$-lineal we mean $(bk)$-convergence ($^1$).

**Corollary 3.** A $K_{bk}$-lineal is a p. c. group.

If the functions $x$ form a linear system and map a certain set of numbers $t$ into a $K_{bk}$-lineal $U$, then they themselves, in a natural way, constitute a $K_{bk}$-lineal $X$ with norm
$\|x\|_X(t) = \|x(t)\|_U$ (if $X$ is $(bk)$-complete).

4. We shall find conditions sufficient for the fulfillment of the relations (1), used below. Let an operation $F$ map a $K_{bk}$-lineal $X$ into a $K_{bk}$-lineal. Denote $F(x + \Delta x) - F(x) = \Delta F(x)$.

**Lemma 1.** If $B$ and $C$ are additive operations, $F$ is uniformly differentiable $((^1), p.372)$, $B \geq F'(x) \geq C$ on $[a, b]_K$, the subset $K \subset X$ is convex, then on all $[x, x + \Delta x]_K \subset [a, b]_K$ the relations

$$B\Delta x \geq \Delta F(x) \geq C\Delta x. \qquad (1)$$

5. We shall establish one sufficient criterion for convergence of an iterative process of Newtonian type, without invoking at this point all the preceding conditions. Suppose that in commutative groups $X$ and $Y$ convergences of countable sequences are defined, with respect to which subtraction is continuous and the limit is unique; convergence in itself, equivalent to convergence, is defined.

$M$ is a subgroup of the group $X_0$, the latter is a subgroup of the group $X$. We induce in $X_0$ and $M$ convergence from $X$, without requiring their closedness with respect to convergence. $K$ is a coset class of $X_0$ modulo the group $M$ with representative $x_0$. The equation

$$P(x) \equiv Dx - F(x) = 0, \tag{2}$$

where the operation $D$ additively maps $X_0$ into $Y$, and $F$ continuously maps $K$ into $Y$. The operation $F$ must also satisfy at least one of the conditions: it admits a continuous extension to some set $K_1 \supset \overline{K}$, or from the convergence $\{x_n\} \subset K$ in $X$ there follows the convergence $\{F(x_n)\}$. The process

$$z_{n+1} = z_n - \Gamma^{-1} P(z_n) \quad (n = 0, 1, 2, \ldots;\ z_0 = x_0), \tag{3}$$

is carried out, where $\Gamma = D - C$, and $C$ is an additive continuous operation defined on $M$, and from the convergence $\{x_n\} \subset K$ in $X$ it follows that $C(x_m - x_n) \to 0$ as $m > n \to \infty$. Suppose, further, that the operation $\Gamma^{-1}$ maps $Y$ into $M$, while $D$ establishes a one-to-one correspondence between $M$ and the image $D(M)$. Thus one may speak of the operation $D^{-1}$, mapping $D(M)$ into $M$.

**Definition 4.** We shall call the image $D(K)$ countably complete if from the relations $z_m, z_n \in K$, $Dz_m - Dz_n \to 0$ as $m > n \to \infty$ there follows the existence of an element $z \in K$ such that $Dz_n \to Dz$.

**Lemma 2.** If the operations $D^{-1}$, $D\Gamma^{-1}$, $\Gamma^{-1}D$ are continuous, the image $D(K)$ is countably complete, and the sequence $z_n$, defined by algorithm (3), converges in $X$, $z_n \to x^*$, then $x^* \in K$, $P(x^*) = 0$.

The simplest case is when $X = Y$ and $D = I$ is the identity operation.

We note that the expression $Dx$ corresponds to the highest derivatives, which enter additively in the differential equation, and such a separation of $Dx$ will make it possible subsequently to restrict the partial ordering only to the functions themselves and to the lower derivatives (see the example in item 9). This extends the class of equations to which the theory of monotone approximation processes is applicable.

6. We give general theorems describing processes of Chaplygin type and alternating approximations (2). Let $X$ be a partially ordered convergence group, $Y$ a partially ordered group, and let all the conditions of the preceding item be satisfied (except for the requirement of convergence of $z_n$);

in fact, in Theorems 1 and 2, instead of $K$ and $M$ only $K_0 = [x_0, \bar{x}_0]_K$ and $M_0 = [0, \xi]_M$ will be used, where $\xi = \bar{x}_0 - x_0$. There are also certain additive operations $A$ and $B$, defined on $M_0$.

**Theorem 1.** Suppose: 1) the operation $\Gamma^{-1}$ is positive ($^1$) and, as $n \to \infty$,

$$A^n \xi \to 0; \tag{4}$$

2) the relations (1) are satisfied for all $x, x + \Delta x \in K_0$ with $\Delta x > 0$; $A \geq \Gamma^{-1}(B - C)$ on $M_0$ and $P(x_0) \leq 0 \leq P(\bar{x}_0)$.

Construct the algorithm

$$x_{n+1} = x_n - \Gamma^{-1} y_n, \quad \bar{x}_{n+1} = \bar{x}_n - \Gamma^{-1} \bar{y}_n \quad (n = 0, 1, 2, ...), \tag{5}$$

where $y_n$ are arbitrary elements satisfying the relations $P(x_n) \leq y_n \leq 0 \leq \bar{y}_n \leq P(\bar{x}_n)$.

Then the algorithms (5) define sequences

$$x_n \leq x_{n+1} \leq x^* \leq \bar{x}_{n+1} \leq \bar{x}_n,$$

enclosing the unique solution $x^*$ of equation (2) on $K_0$. If $y_n = P(x_n)$, then both approximations converge to the solution with rate

$$|x_n - x^*| \leq A^n \xi.$$

The algorithms (5) can be modified by replacing the operation $\Gamma$ by the operation $\Gamma_n = D - C_n$, where the additive operation $C_n \geq C$ is such that $\Gamma_n^{-1} > 0$ and $C_n \Delta x \leq \Delta F(x)$ on $[x_n, \bar{x}_n]_K$ for $\Delta x > 0$.

7. Let all the conditions specified in the preceding item before the formulation of Theorem 1 be satisfied. The notation $\bigvee$ has one of the meanings $\leq, \geq$.

**Theorem 2.** Let condition 1) of Theorem 1 be satisfied and $C\Delta x \geq \Delta F(x) \geq B\Delta x$ for $x, x + \Delta x \in K_0$, $\Delta x > 0$; $A \leq \Gamma^{-1}(B - C)$; $-P(x_0) \leq \Gamma(\bar{x}_0 - x_0) \geq P(\bar{x}_0)$. In one of the algorithms (5), take such $y_n$ that

$$\Gamma(x_n - x_{n-1}) \bigvee y_n \bigvee P(x_n),$$

where $x_0 = x_0$ (or $\bar{x}_0$), $x_{-1} = \bar{x}_0$ (or $x_0$).

Then

$$x_{2n} \bigvee x_{2n+2} \bigvee x^* \bigvee x_{2n+1} \bigvee x_{2n+1},$$

where $x^*$ is the unique solution of equation (2) on $K_0$. If $y_n = P(x_n)$, then $x_n \to x^*$ with rate

$$|x_n - x^*| \leq (-A)^n |x_1 - x_0| \leq (-A)^n \xi.$$

If $x_n \neq x^*$, then it is possible to choose $y_n \neq \Gamma(x_n - x_{n-1})$ and continue the process.

8. Consider a system of ordinary differential equations of Volterra type (³) in matrix notation

$$z'(t) = f(t, u, v), \quad u = z(t), \quad v = z(s), \quad s \in E_t, \quad \sup E_t \leq t \in [0, T],$$

$$z(s) = \varphi(s) \quad \text{for } s \leq 0,$$

where $\varphi$ is continuous, and $f$ is continuous in $t$ and continuously differentiable in $u$ and $v$. Put $x(t) = z'(t)$ and take the linear space $C([0, T])$ of all continuous vector-functions $x$ as $X = X_0 = M = K = Y$. Put $D = I$, $F(x) = f(t, u, v)$. Let

$$B_1 \geq \frac{\partial f}{\partial u} \geq C_1, \quad B_2 \geq \frac{\partial f}{\partial v} \geq C_2 \leq 0,$$

where the matrices of the linear algebraic transformations $B_i, C_i$ are composed of constant coefficients. If one sets

$$B\Delta x = (B_1 + B_2) \int_0^t \Delta x(\tau)\, d\tau, \qquad C\Delta x = (C_1 + C_2) \int_1^t \Delta x(\tau)\, d\tau,$$

then relations (1) and (4) are satisfied. If $\Gamma^{-1} > 0$, then the remaining requirements of item 6 concern only the initial approximations $x_0$ and the residuals $P(x_0)$. The situation is analogous for Theorem 2.

9. A plane closed domain $E$ is bounded "from below and from the left" by the graph $l$ of the function $t = \mu(s)$ $(s = \nu(t))$, which has a continuous derivative $\mu'(s) < 0$, and from above and from the right by horizontal and vertical straight lines intersecting $l$. On $E$ the equation $u_{st} = f(s, t, u, v, w)$, $v = u_s$, $w = u_t$, is given, with boundary conditions (⁴)

$$u|_l = \varphi(s, t) \equiv \Phi(s), \qquad w|_l = \Psi(s), \qquad v|_l = \Phi'(s) - \Psi(s)\mu'(s).$$

   Let $f$ be continuous in $s, t$ and have continuous derivatives with respect to $u, v, w$.

As $X$ take the linear space of all pairs $x = (v, w)$ of continuous functions $v(s, t), w(s, t)$ defined on $E$; convergence is uniform. We single out the subgroup $X_0$ consisting of all those pairs $x$ of continuous functions $v, w$ which have continuous derivatives $v_t, w_s$ and whose integrals on $E$ coincide:

$$\int_{s_0}^{s} v(\sigma, t)\, d\sigma = \int_{t_0}^{t} w(s, \tau)\, d\tau. \tag{6}$$

It follows from the definition of $X_0$ that $v_t \equiv w_s$ for all $(v, w) \in X_0$. We form the group $M$ and the class $K$ from $(v, w) \in X_0$ satisfying, respectively, the zero and the given boundary conditions on $l$ for $v$ and $w$. The role of the set $K_1$ is played by a set differing from $K$ only in that the pair $(v, w)$ is not required to have continuous derivatives $v_t, w_s$. The elements of the group $M$ are denoted by $\Delta x = (\Delta v, \Delta w)$. As $Y$ we take the linear space $C(E)$ of all continuous functions with uniform convergence. The role of the operations $D$ in $X_0$ and $F$ in $K$ is played by $Dx = D(v, w) = v_t \equiv w_s$, $F(x) = f(s, t, u, v, w)$, where the difference $u(s, t) - \varphi(s_0, t_0)$ is equal to the second of the integrals (6).

Suppose there exist such functions $u_0, \overline{u}_0$, satisfying the given boundary conditions and having continuous derivatives $v_0, w_0, D(v_0, w_0)$, for which on $E$ the inequalities $v_0 \leq \overline{v}_0$, $w_0 \leq \overline{w}_0$, $P(x_0) \leq 0 \leq P(\overline{x}_0)$ hold. Suppose, further, that for all $u$ (with continuous first-order derivatives) such that $u|_l = \Phi(s_0)$, $v_0 \leq u_s \leq \overline{v}_0$, $w_0 \leq u_t \leq \overline{w}_0$ (and, consequently, $u_0 \leq u \leq \overline{u}_0$), the inequalities

$$a_0 \geq \partial f/\partial v \geq a \geq 0, \qquad b_0 \geq \partial f/\partial w \geq b \geq 0, \qquad c_0 \geq \partial f/\partial u \geq -ab$$

hold. We take $B\Delta x = a_0\Delta v + b_0\Delta w + c_0\Delta u$, $C\Delta x = a\Delta v + b\Delta w - ab\Delta u$. Then, as the operation $A$, one may take

$$A\Delta x = (A_1\Delta v, A_2\Delta w), \qquad A_1 = \int_{s_0}^{s}\left[\alpha\cdot\sigma(t) + \beta\int_{t_0}^{t}\cdot(\sigma, \tau)\,d\tau\right]d\sigma.$$

$A_2$ has an analogous form; $\alpha, \beta > 0$ are certain numbers. All the conditions of Theorem 1 are fulfilled. The algorithm

$$u_n - u_{n+1} = \int_{t_0}^{t}\int_{s_0}^{s} e^{a(t-\tau)+b(s-\sigma)} y_n(\sigma, \tau)\,d\sigma\,d\tau,$$

the choice of $y_n$ was indicated above.

Gorky State University
named after N. I. Lobachevsky

Received
19 V 1962

## CITED LITERATURE

[1] B. Z. Vulikh, *Introduction to the Theory of Partially Ordered Spaces*, 1961.
[2] S. N. Slugin, DAN, **136**, No. 1, 36 (1961).
[3] A. N. Tikhonov, *Bull. Moscow State Univ.*, A, **1**, 8 (1938).
[4] S. L. Sobolev, *Equations of Mathematical Physics*, 1954.