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Abstract

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ON SYSTEMS OF NONSPECIAL SUBGROUPS OF FINITE GROUPS

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§ 1. In the present paper we continue the study of systems of nonspecial subgroups introduced by us earlier in ([1]). In doing so, we introduce the general notion of an oriented subgroup and of an oriented Π-system, generalizing the notion, introduced in ([1]), of an oriented ΠS-system, and prove a theorem on the existence of oriented Π-systems in nonspecial groups (Theorem 1). We also investigate the influence of the dispersiveness of a group on the existence in the group of systems of subgroups of a definite kind (Theorems 2, 3, and 4). Along the way we obtain a theorem on the factorization of solvable groups, based on the decomposition, introduced by us, of the set of prime divisors of the order of a group into the so-called S-classes (Theorem 6). We also introduce indecomposable Π-systems and establish a theorem on their existence.

§ 2. We shall use the notation and definitions introduced in ([1]), as well as the following. $\Pi(n)$, where n is a natural number, is the set of all prime divisors of n. $\mathfrak{G}$ is a finite group of order $(\mathfrak{G})$; when $(\mathfrak{G}) = 1$, we put $\mathfrak{G} = \mathfrak{E}$. Π is some empty or nonempty subset of the set $\Pi((\mathfrak{G}))$; k is the number of elements of Π. A Π-group is a finite group $\mathfrak{G}$ for which $\Pi = \Pi((\mathfrak{G}))$. By a Π-system of subgroups of the group $\mathfrak{G}$ (more briefly, a Π-system of the group $\mathfrak{G}$) we shall mean, for nonempty Π, such a set K of pairwise nonisomorphic subgroups of the group $\mathfrak{G}$ that there exists a one-to-one mapping φ of the set Π onto the set $K = \varphi(\Pi)$, under which the image $\varphi(p_i)$ of each $p_i \in \Pi$ is a $p_i d$-subgroup of $\mathfrak{G}$. We shall call the function φ the defining function of the system K. For empty Π, by definition, we shall regard the empty set as a Π-system $\varphi(\Pi)$ of the group $\mathfrak{G}$. A Π-system for $\Pi = \Pi((\mathfrak{G}))$ will simply be called a system. If $p_i \in \Pi$, then a $p_i d$-subgroup $\mathfrak{H}$ of the group $\mathfrak{G}$ will be called p_i-oriented relative to the group $\mathfrak{G}$ if $\mathfrak{H}$ inherits from $\mathfrak{G}$ the properties of being non-p_i-nilpotent and non-p_i-decomposable. A Π-system $\varphi(\Pi)$ of the group $\mathfrak{G}$ will be called an oriented Π-system of the group $\mathfrak{G}$ if Π is empty or if for every $p_i \in \Pi$ the subgroup $\varphi(p_i)$ is p_i-oriented relative to $\mathfrak{G}$. If the group $\mathfrak{G}$ is a group of type S or a direct product of two groups of coprime orders, one of which is of type S and the other cyclic of prime order, then $\mathfrak{G}$ will be called a group of type SC (or an SC-group). A Π-system (oriented Π-system) will be called: a ΠS-system (oriented ΠS-system) if Π is empty or if for every $p_i \in \Pi$ the subgroup $\varphi(p_i)$ is of type S; a ΠSC-system (oriented ΠSC-system) if Π is empty or if for every $p_i \in \Pi$ the subgroup $\varphi(p_i)$ is of type SC. When $\Pi = \Pi((\mathfrak{G}))$ we shall omit the letter Π in these designations. A dispersive group is a group $\mathfrak{G}$ that is σ-

dispersive ([2]), where σ is the set $\Pi((\mathfrak{G}))$ endowed with some ordering (cf. Ore ([3])). By a group of type $Z_p^{(1)}$ we shall mean a p-indecomposable group whose order is divisible by no more than three distinct primes and which contains no more than one class of nontrivial isomorphic p-indecomposable ⋯

subgroups. A Π-set $\varphi(\Pi)$ will be called an indecomposable Π-set if Π is empty or if, for every $p_i \in \Pi$, the subgroup $\varphi(p_i)$ is of type $Z_{p_i}^{(1)}$.

It is obvious that the class of groups of type SC is the set of all groups of type S_p, as p ranges over all prime numbers.

§ 3. We present the results obtained by us concerning sets of subgroups.

Theorem 1. A nonspecial group $\mathfrak{G}$ has at least one oriented ΠSC-set containing no fewer than $k-1$ subgroups.

Theorem 2. If $\mathfrak{G}$ has no ΠSC-set of nontrivial subgroups containing k subgroups, then $\mathfrak{G}$ is a dispersion group.

Theorem 3. Let $\mathfrak{G} \neq \mathfrak{E}$ not decompose into a direct product of nontrivial subgroups of pairwise relatively prime orders. Then: 1) if $\mathfrak{G}$ is not a dispersion Π-group, then it has at least one ΠS-set containing k subgroups; 2) if $\mathfrak{G}$ is a dispersion Π-group, then it has at least one ΠS-set containing $k-1$ subgroups.

Theorem 4. Suppose that for each $p_i \in \Pi$ the group $\mathfrak{G}$ is indecomposable, and suppose that in the decomposition of $\mathfrak{G}$ into a direct product of nontrivial subgroups of pairwise relatively prime orders, each of which is already indecomposable in this same way, there are altogether l direct factors that are dispersion Π-groups.

Then $\mathfrak{G}$ has at least one ΠS-set containing $k-l$ subgroups.

Theorem 5. If for every $p_i \in \Pi$ the group $\mathfrak{G}$ is not p_i-decomposable and if $\mathfrak{G}$ is not of type S, then it has at least one indecomposable Π-set containing k subgroups.

§ 4. For the proof of Theorem 3 (and of the following Theorem 4 derived from it), we use the notion, introduced below, of S-connectedness of the prime divisors of the order of a group and the resulting factorization of soluble groups (Theorem 6). We give the main points of this method.

Let a symbol of the form $\mathfrak{S}_i(p,q) = \mathfrak{S}_i(q,p)$ denote some subgroup of type S, whose order is generated by the prime numbers p and q. Then the prime divisors p and q of the order of the group $\mathfrak{G}$ will be called S-connected in the group $\mathfrak{G}$ by means of the sequence of prime numbers

$$p^{(1)},\ p^{(2)}, \dots,\ p^{(t+1)}, \tag{1}$$

if in $\mathfrak{G}$ there exists a sequence of subgroups of type S that can be written in the form

$$\mathfrak{S}_1(p^{(1)}, p^{(2)}),\ \mathfrak{S}_2(p^{(2)}, p^{(3)}),\ \mathfrak{S}_3(p^{(3)}, p^{(4)}), \dots, \mathfrak{S}_t(p^{(t)}, p^{(t+1)}), \qquad (2)$$

where $p^{(1)} = p$ and $p^{(t+1)} = q$, $t \geq 1$, and both sequence (1) and sequence (2) may have repetitions. We shall call sequence (2) an S-chain passing through sequence (1).

Lemma. If Π is the set formed by all members of sequence (1) except the last, then $\mathfrak{G}$ has a ΠS-set containing k subgroups.

Let now M be the set of all those elements of $\Pi((\mathfrak{G}))$ with respect to each of which $\mathfrak{G}$ is not decomposable(1). By means of Theorem 2 from (1) it is not difficult to see that, if M is nonempty, then the property of S-connectedness partitions M into classes $M_1, M_2, \dots, M_r$ of mutually S-connected numbers, which we shall call the S-classes of the group $\mathfrak{G}$.

By Π_i we shall denote some subset of the set $\Pi((\mathfrak{G}))$, and by $\mathfrak{G}_{\Pi_i}$ the set of all Π_i-elements of $\mathfrak{G}$.

Theorem 6. If $\mathfrak{G} \neq \mathfrak{E}$ is a soluble group, then

$$\mathfrak{G} = \mathfrak{G}_{\Pi_1} \times \mathfrak{G}_{\Pi_2} \times \cdots \times \mathfrak{G}_{\Pi_\omega},$$

where Π_i, $i = 1, 2, \dots, \omega$, either consists of only one prime number, or is an S-class of the group $\mathfrak{G}$, and $\mathfrak{G}_{\Pi_i}$ is a subgroup which is no longer decomposable into a direct product of nontrivial subgroups of pairwise coprime orders.

§ 5. Theorem 4 raises the lower bound for the number of subgroups in ΠS-sets of a group, found by V. I. Sergienko (4).

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Note: Figure translations are in progress. See original paper for figures.