Hydromechanics

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Abstract

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THE EFFECT OF A MAGNETIC FIELD ON SURFACE WAVES IN A CONDUCTING LIQUID

In the present work we consider the effect of an external magnetic field on the process of propagation of gravity-capillary waves in a conducting liquid.

Choose the z-axis in the direction of the gravitational field and assume that the half-space $z < 0$ is filled with an ideal incompressible liquid situated in an external uniform magnetic field $\mathbf{H}_0$ and in the field of gravity. The conductivity of the liquid is assumed to be infinite. It will be shown below that, in the wave motion of a conducting liquid, an additional variable field $\mathbf{h}$ is generated. If the wave amplitude a is much smaller than the wavelength λ, then [1] in the equations of motion of the medium one may neglect the term $(\mathbf{v}\nabla)\mathbf{v}$ in comparison with $\partial\mathbf{v}/\partial t$. It turns out that this same condition is sufficient for the magnetic field $\mathbf{h}$ to be much smaller than the external field $\mathbf{H}_0$. Under these assumptions the equations of magnetic hydrodynamics [2] are linearized, and they may be written in the form

$$\partial\mathbf{v}/\partial t = -\frac{1}{\rho}\nabla p + \mathbf{g} + \frac{1}{4\pi\rho}[\operatorname{rot}\mathbf{h}\mathbf{H}_0], \qquad \partial\mathbf{h}/\partial t = \operatorname{rot}[\mathbf{v}\mathbf{H}_0], \tag{1}$$

$$\operatorname{div}\mathbf{v} = 0, \qquad \operatorname{div}\mathbf{h} = 0.$$

The electric field inside the liquid, after solving system (1), can be found [2] from

$$\mathbf{E} = -\frac{1}{c}[\mathbf{v}\mathbf{H}_0].$$

The boundary conditions for these equations are the continuity, at the surface of the liquid, of the magnetic and electric fields, and also of the pressure.

We shall seek a periodic solution of system (1), i.e., assume that $\mathbf{v}$ and $\mathbf{h}$ are proportional to $e^{-i\omega t}$.

After some transformations, system (1) can be rewritten in the form

$$\operatorname{rot}\left\{\omega^2\mathbf{v}+\frac{1}{4\pi\rho}(\mathbf{H}_0\nabla)^2\mathbf{v}\right\}=0,\qquad \operatorname{div}\mathbf{v}=0,\qquad -i\omega\mathbf{h}=(\mathbf{H}_0\nabla)\mathbf{v}. \tag{2}$$

The equation $\operatorname{div}\mathbf{h}=0$ is satisfied automatically. From (2) it is evident that one can introduce a function φ, which is a certain analogue of the velocity potential, $\mathbf{v}=\nabla\varphi$, and is defined by the formula

$$\nabla\varphi=\omega^2\mathbf{v}+\frac{1}{4\pi\rho}(\mathbf{H}_0\nabla)^2\mathbf{v}. \tag{3}$$

Substitution of (3) into (2) gives

$$\Delta\varphi=0. \tag{4}$$

Below we restrict ourselves to the case in which the external field $\mathbf{H}_0$ is directed along the z-axis. We seek the solution of the last equation in the form

$$\varphi=Ce^{ikx+kz}. \tag{5}$$

We orient the x-axis in the direction of wave propagation. Then (3) becomes

$$\omega^2\mathbf{v}+\frac{H_0^2}{4\pi\rho}\frac{\partial^2\mathbf{v}}{\partial z^2}=C\nabla e^{ikx+kz}. \tag{6}$$

A particular solution of (6) for the velocity components is

$$v_x=\frac{4\pi\rho ki}{4\pi\rho\omega^2+k^2H_0^2}Ce^{ikx+kz},\qquad v_y=0,\qquad v_z=\frac{4\pi\rho k}{4\pi\rho\omega^2+k^2H_0^2}Ce^{ikx+kz}. \tag{7}$$

This solution describes surface waves and coincides in form with the solution in the absence of an external field $\mathbf{H}_0$. To the solution (7), however, one must also add a solution of the homogeneous equation

$$\omega^2\mathbf{v}+\frac{H_0^2}{4\pi\rho}\frac{\partial^2\mathbf{v}}{\partial z^2}=0. \tag{8}$$

The latter describes an Alfvén wave propagating into the depth of the liquid. Taking into account that the time dependence has already been explicitly separated in the form $e^{-i\omega t}$, we obtain

$$\mathbf{v}=\mathbf{B}e^{-i(pz+\omega t)}, \tag{9}$$

where $p = 4\pi\rho\omega^2/H_0^2$.

Because of the periodicity of all processes occurring in the wave, $\mathbf{B}$ should be taken in the form $\mathbf{B} = \mathbf{B}_0 e^{ikx}$, where $\mathbf{B}_0$ is a constant vector. In addition, the solution (9) must satisfy $\operatorname{div}\mathbf{v} = 0$. Hence we obtain one relation between the components of $\mathbf{B}_0$,

$$kB_{0x} - pB_{0z} = 0. \tag{10}$$

The remaining two unknown components of the vector $\mathbf{B}_0$ must be determined from the boundary conditions. For this it is necessary to find the electromagnetic field in the half-space above the liquid. For the vector potential of the electromagnetic field $\mathbf{A}$ we have the equation

$$\Delta\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = 0. \tag{11}$$

Since the electric and magnetic fields arising in the liquid are proportional to $e^{i(kx-\omega t)}$, it is necessary to require that

$$\mathbf{A}\Big|_{z=0} = \mathbf{A}_0 e^{i(kx-\omega t)}. \tag{12}$$

A solution of equation (11) satisfying the boundary condition (12) is

$$\mathbf{A} = \mathbf{A}_0 e^{i(kx-\omega t)-\chi z}, \tag{13}$$

where $\chi^2 = k^2 - \omega^2/c^2$, and $\mathbf{A}_0$ is a constant vector. Thus the electromagnetic field is localized in the space above the liquid in a layer whose thickness is approximately equal to the surface wavelength λ. The electric and magnetic fields must be continuous at the surface of the liquid. In view of the condition $a \ll \lambda$, one may approximately assume that the condition of continuity of the fields must be satisfied in the plane $z = 0$. Knowing the distribution of velocities inside the liquid (7) and (9), we determine, using (2) and (3), the electric and magnetic fields which must coincide at $z = 0$ with the fields obtained from the solution (13). Taking into account that $\chi^2 \simeq k^2$, we find

$$A_{0x} = 0, \qquad A_{0y} = \frac{4\pi\rho k}{4\pi\rho\omega^2 + k^2 H_0^2}\, C\,\frac{H_0}{\omega}, \qquad A_{0z} = 0, \qquad \mathbf{B}_0 = 0. \tag{14}$$

From (14) and (7) we conclude that the liquid particles in the wave describe circles with a radius that decreases exponentially in the direction into the depth of the liquid. It remains to establish the dispersion law for the waves under consideration.

From (1), for the pressure in the liquid we obtain

$$p = p_0 + \frac{i\omega\rho}{k} v_z - g\rho z; \tag{15}$$

p_0 is a pressure constant (including the static part of the pressure associated with the constant field $\mathbf{H}_0$). The pressure above the surface of the liquid is repre-

we write in the form

$$p = p_\mathrm{r} + T_{zz}, \tag{16}$$

where p_r is the gas pressure above the liquid and

$$T_{zz} = \frac{1}{4\pi}\left\{-E_z^2 - H_z^2 + \frac{1}{2}(E^2 + H^2)\right\} \tag{17}$$

is a component of the Maxwell stress tensor.

From (3) we have $h \sim vH_0/\omega\lambda$, but $v/\omega \sim a$, whence $h/H_0 \sim a/\lambda$, so that indeed

$$h/H_0 \ll 1, \tag{18}$$

which justifies the initial assumption.

On the basis of (16) and (17), taking (18) into account, the pressure outside the liquid may be written as

$$p \simeq p_0 - \frac{H_0 h_z}{4\pi}. \tag{19}$$

At the surface of the liquid, (15) must coincide with (19). Denoting the displacement of the liquid particles from the equilibrium position at the surface by ζ, we obtain the condition:

$$\frac{i\omega\rho}{k} v_z\Big|_{z=\zeta} - g\zeta\rho = \frac{H_0}{4\pi i\omega}\frac{\partial}{\partial z} v_z\Big|_{z=\zeta}. \tag{20}$$

Since $a \ll \lambda$, it may be assumed that $v_z = \partial\zeta/\partial t$. By virtue of this same condition, to within quantities of second order of smallness, (20) must be satisfied at $z = 0$. Differentiating (20) with respect to t and canceling $v_z|_{z=0}$, we obtain

$$\frac{\omega\rho}{k} - g\rho = -\frac{H_0^2}{4\pi}k. \tag{21}$$

Relation (21) represents the dispersion law for magnetohydrodynamic gravitational waves. For $H_0 = 0$ we obtain, as was to be expected, the usual dispersion law $\omega = \sqrt{kg}$.

It follows from (21) that in a magnetic field the propagation of gravitational waves of arbitrary length is impossible. Indeed, from (21) and the condition $\omega^2 > 0$ (a traveling wave with non-decaying amplitude) we have the condition imposed on $\lambda = 1/k$:

$$\lambda > H_0^2/4\pi\rho g. \tag{22}$$

Gravitational waves with λ smaller than $\lambda_{\text{cr}} = H_0^2/4\pi\rho g$ cannot propagate along the surface of the liquid.

Let us consider the influence of surface tension under the assumption that the surface tension does not depend on the fields, and that the electromagnetic stresses, given by the components of the tensor $T_{\alpha\beta}$, are small in comparison with the stresses due to the change in curvature of the surface. Then it is sufficient simply to add $-\alpha k^2$ to the left-hand side of (21), and the dispersion law takes the form

$$\omega^2 - \frac{\alpha k^3}{\rho} - gk = -\frac{H_0^2}{4\pi}k^2 \tag{23}$$

(here α is the coefficient of surface tension). The condition for positivity of ω^2, as follows from (23), is

$$\alpha k^2 + g\rho - \frac{H_0^2}{4\pi}k > 0. \tag{24}$$

From (24) it is seen that in fields so weak that

$$H_0^2 < 8\pi\sqrt{\alpha g\rho}, \tag{25}$$

waves of any length with the dispersion law (23) are possible. If, however, the inequality opposite to (25) holds, then condition (24) is satisfied only for $k_1 < k < k_2$, where

$$k_{1,2} = H_0^2/8\pi\alpha \mp \sqrt{(H_0^2/8\pi\alpha)^2 - g\rho/\alpha}. \tag{26}$$

Thus, without damping, only sufficiently long or sufficiently short waves can propagate, and the dispersion law does not depend on the orientation of the magnetic field relative to the z-axis.

Let us note that from (7) and (2) it follows that $\Delta\mathbf{h} = 0$; therefore the term containing the magnetic viscosity ν_m drops out automatically even without the assumption $\sigma = \infty$. This means that, for the type of motion considered, magnetic viscosity should be taken into account only together with hydrodynamic viscosity, or in subsequent approximations in a/λ. Since for mercury, for example, $\nu \sim 10^{-3}$ cm^2/sec, one may take $\sigma = \infty$ (i.e., $\nu_m = 0$) down to very low frequencies. The calculation of the damping of surface waves in the opposite limiting case was carried out in ([3]).

We shall not consider here the question of how the damping of waves whose lengths lie in the forbidden region occurs. We note only that, as λ approaches λ_{cr}, ω tends to 0, the motion is characterized by the Reynolds number $\mathrm{Re} < 1$, and the liquid can no longer be regarded as ideal, nor its conductivity as infinite.

The results obtained have a clear physical interpretation. Since under the condition $\sigma = \infty$ each liquid particle in the wave is rigidly bound to a definite magnetic-field line and at the same time describes a circle of radius $\sim a$, the wave motion gives rise to a field $\mathbf{h}$, bending in the corresponding way the lines of force of the external magnetic field $\mathbf{H}_0$. The wave circular motion of the particles penetrates into the depth of the liquid over distances $\sim \lambda$; therefore $h \sim \dfrac{a}{\lambda} H_0$, while the energy of the resulting additional magnetic field per unit length of the wave front is of order $h^2\lambda \sim \dfrac{a^2}{\lambda} H_0^2$. The mechanical energy of the wave per unit front is of order $\rho v^2 \lambda \sim \rho a^2 \omega^2 \lambda$, if one takes into account that $v \sim a\omega$. But for the existence of waves it is necessary that the liquid particles be able to describe a complete circumference, i.e., that the mechanical energy suffice to create the corresponding field $\mathbf{h}$. Hence we obtain $\dfrac{a^2}{\lambda} H_0^2 < \rho a^2 \omega^2 \lambda$, or

$$H_0^2/\rho < \omega^2/k^2. \tag{27}$$

Substituting here $\omega = \sqrt{gk}$, we obtain a restriction of type (22) on the admissible length of gravitational waves. From (27) it also follows, in accordance with (26), that propagation of sufficiently short waves is always possible, since for capillary waves $\omega^2 = \alpha k^3/\rho$.

From what has been said it is clear that the qualitative results of the work are not connected with the orientation of the external magnetic field. This conclusion is confirmed by direct calculation. Thus, in ([4]) it was shown that a longitudinal external magnetic field also has a stabilizing effect on surface disturbances. In ([5]) the damping of surface waves by an external magnetic field perpendicular to the undisturbed surface of the liquid was directly observed.

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Note: Figure translations are in progress. See original paper for figures.