MATHEMATICS

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Abstract

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ON A CLASS OF CONGRUENCES OF CURVES OF THE SECOND ORDER WITH A DEGENERATING FOCAL SURFACE

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The paper considers congruences of curves of the second order ([1]) in three-dimensional projective space, possessing the following properties: 1) one focal surface of the congruence degenerates into a point; 2) there exist two nondevelopable focal surfaces (B_{-1}) and (B_1); 3) the focal surfaces (B_{-1}) and (B_1) are not the envelopes of the planes of the conics of the congruence; 4) the focal lines of each of these surfaces, together with the lines conjugate to them, form nets R ([2]), which are the second Laplace transform of one another. We shall agree to call these congruences C_1-congruences.

§ 1. **Derivation formulas of the canonical frame. Focal surfaces and focal families of a C_1-congruence.** Choose the vertices B_{-1} and B_1 of the frame $T(B_{-1}, B_0, B_1, A_2)$ at points describing the nets R on the focal surfaces (B_{-1}), (B_1), the vertex B_0 in their common first Laplace transform, and, finally, identify the vertex A_2 with the fixed focal point of the congruence C_1. With a certain normalization of the vertices, the derivation formulas of the canonical frame take the form:

$$dB_{-1} = -b\omega^2 P_0 + a\omega^1 P_1, \qquad dB_1 = \omega^1 P_0 + \omega^2 P_2, \qquad dA_2 = 0, \qquad (1)$$

$$\alpha B_0 = -\omega^2 B_{-1} + \frac{1}{2}\left\{(1+ha^2)\omega^1 + \left(a - 2b - \frac{1}{ha}\right)\omega^2\right\} B_0 - ha^2 b\omega^1 B_1,$$

where

$$P_0 = B_{-1} + B_1 - A_2, \qquad P_1 = B_0 + haB_{-1}, \qquad P_2 = B_0 - aB_1. \qquad (2)$$

Here ω^1 and ω^2 are Pfaffian forms satisfying the structural equations

$$D\omega^1 = (a-b)[\omega^1\omega^2], \qquad D\omega^2 = \frac{1}{2}(1-ha^2)[\omega^1\omega^2], \qquad (3)$$

$h \neq 0$ is an arbitrary constant, and the invariants a and b satisfy the system of two ordinary differential equations

$$2\frac{d\ln a}{du} = 1 + ha^2 - 2hab, \qquad 2\frac{d\ln b}{du} = 3(ha^2 - 1), \tag{4}$$

where $du = \omega^1 + \frac{1}{ha}\omega^2$ is a complete differential.

Consequently, the C_1-congruences exist and are determined up to a choice of three constants.

The conic of the C_1-congruence with respect to the canonical frame is determined by the equations

$$x^1x^{-1} + x^{-1}x^2 + x^2x^1 = 0, \qquad x^0 = 0. \tag{5}$$

The foci, distinct from B_{-1}, B_1, and the focal families that are not the lines $\omega^1\omega^2 = 0$, are found from equations (5) and the equations

$$a\omega^1x^{-1} + \omega^2x^1 = 0, \qquad \omega^2(a - 2b)x^{-1} + (2 - ha^2)\omega^1x^1 = 0. \tag{6}$$

Theorem 1. The congruence C_1 has four nondegenerate focal surfaces and four focal families.

§ 2. Geometric properties of the congruence C_1.

Denote by B_{-n}, B_n the n-th Laplace transform of the surface B_0, respectively in the direction of the lines $\omega^1 = 0$, $\omega^2 = 0$. For any natural number n, the following formulas hold:

$$\begin{aligned}
B_2 &= B_{-1} - A_2, & B_{-2} &= A_2 - B_1,\\
B_{3n} &= B_0 + ahP_{n-1}(h)A_2, & B_{-3n} &= B_0 - ah^{1-n}P_{n-1}(h)A_2,\\
B_{1+3n} &= B_1 + hP_{n-1}(h)A_2, & B_{-(1+3n)} &= B_{-1} + h^{-n}P_{n-1}(h)A_2,\\
B_{2+3n} &= B_{-1} - P_n(h)A_2, & B_{-(2+3n)} &= B_1 - h^{-n}P_n(h)A_2,
\end{aligned} \tag{7}$$

where $P_n(h) = h^n + h^{n-1} + \dots + h + 1$.

From formulas (7) the following theorems follow:

Theorem 2. If the natural numbers n and $n+1$ are not congruent to the number m modulo 3 ($m = 0, 1, 2$), then the line $B_{nB_{n+1}}$ passes through the point P_m.

Theorem 3. If k and $k-1$ are negative integers such that the nonnegative numbers $3n+k$ and $3n+k-1$ (n natural) are not congruent to the number m modulo 3 ($m = 0, 1, 2$), then the line $B_{kB_{k-1}}$ passes through the point P_m.

Theorem 4. If, for an arbitrary integer k, we have

$$k \equiv p \pmod 3, \qquad p = 0,\ \pm 1,\ \pm 2,$$

then the Laplace transform B_k lies on the line AB_p.

Theorem 5. The cross ratios $(B_0A;\, B_{3n}B_{-3n})$, $(B_1A;\, B_{1+3n}B_{1-3n})$, $(B_{-1}A;\, B_{-1+3n}B_{-1-3n})$ are equal, with the opposite sign, to the n-th power of the invariant h.

Analyzing the properties of the three-vertex figure P_0, P_1, P_2, we see that its sides touch the lines $\omega^1 = 0$ and $\omega^2 = 0$ at the corresponding vertices. The plane (P_0, P_1, P_2) is stationary. The lines $\omega^1 = 0$, $\omega^2 = 0$, $\omega^1 + \frac{1}{ha}\omega^2 = 0$ form on it three nets described by the vertices P_0, P_1, and P_2.

§ 3. Some subclasses of congruences C_1.

1) **Congruences $h = -1$.** They are determined with arbitrary two constants and are characterized by the fact that $B_{-3} = B_3$, i.e., the Laplace sequence $\{B_{-p}, B_q\}$ is closed with period 6.
2) **Congruences $h = 1$.** They are also determined with arbitrary two constants and are characterized by the fact that all quadruples of points indicated in Theorem 5 are harmonic. The asymptotic lines of one family of all surfaces R of the congruence $h = 1$ are lines with constant invariants and are mapped by the rays of rectilinear congruences onto the lines of the stationary plane (P_0, P_1, P_2) passing through the fixed focal point of the congruence C_1.
3) **Congruences $a = b = h^{-1/2}$.** They are determined with arbitrary one constant and are the only congruences C_1 with constant invariants.

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CITED LITERATURE

1. N. G. Tuganov, *DAN*, **100**, No. 1 (1955).
2. S. P. Finikov, *Theory of Congruences*, Moscow–Leningrad, 1950.

Note: Figure translations are in progress. See original paper for figures.