

INVERSE SEMIGROUPS WITH A FINITE NUMBER OF IDEMPOTENTS

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Abstract

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MATHEMATICS

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INVERSE SEMIGROUPS WITH A FINITE NUMBER OF IDEMPOTENTS

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Let \mathfrak{K} denote the class of inverse semigroups with a finite number of idempotents. To simplify the formulations we shall assume that every semigroup in \mathfrak{K} contains a zero. For semigroups of the class \mathfrak{K} we shall briefly describe: their structure, the automorphism group, the semigroup of related one-sided shifts⁽¹⁾, homomorphisms, and the construction procedure. The latter, together with the isomorphism theorem (Theorem 2), also gives at the same time a description of semigroups of the class \mathfrak{K} .

Let \mathfrak{E} denote the class of completely simple semigroups whose defining matrix is finite and may be chosen to be the identity matrix. Everywhere below \mathfrak{A} denotes a semigroup with zero 0. We shall write $A \rightarrow B$ if B follows from A .

1. The foundation of the theory set forth is

Theorem 1. \mathfrak{A} belongs to the class \mathfrak{K} if and only if \mathfrak{A} has a finite ideal composition series

$$\mathfrak{A} = \mathfrak{A}_1 \supset \mathfrak{A}_2 \supset \dots \supset \mathfrak{A}_n \supset \mathfrak{A}_{n+1} = 0, \quad (1)$$

all of whose factors belong to \mathfrak{E} . In particular, $\mathfrak{E} \subset \mathfrak{K}$.

1, A. Everywhere (1) will denote an ideal composition series of \mathfrak{A} ; the factors of (1) will be denoted by $\overline{\mathfrak{A}}_i$ ($i = 1, 2, \dots, n$), $\overline{\mathfrak{A}}_i = \mathfrak{A}_i / \mathfrak{A}_{i+1}$.

2. Let P be a field, and let $P(\mathfrak{B})$ be the semigroup algebra of the semigroup \mathfrak{B} over P^* .

To the series (1) there corresponds the ideal series $P(\mathfrak{A})$

$$P(\mathfrak{A}) = P(\mathfrak{A}_1) \supset P(\mathfrak{A}_2) \supset \dots \supset P(\mathfrak{A}_n) \supset P(\mathfrak{A}_{n+1}) = 0. \quad (2)$$

Here

$$P(\mathfrak{A}_i) / P(\mathfrak{A}_{i+1}) \simeq P(\overline{\mathfrak{A}}_i) \quad (i = 1, 2, \dots, n).$$

Let α_i denote the ideal homomorphism of \mathfrak{A} generated by \mathfrak{A}_i , a term of (1); the induced homomorphism of $P(\mathfrak{A})$ by α_i will also be denoted by α_i . Then

$$P(\overline{\mathfrak{A}}_i) = \alpha_{i+1}P(\mathfrak{A}_i) \quad (i = 1, 2, \dots, n).$$

Let E_i denote the sum in $P(\mathfrak{A})$ of all idempotents from

$$\mathfrak{A}_i \setminus \mathfrak{A}_{i+1} \quad (i = 1, 2, \dots, n).$$

Let now $k \leq i \leq n$. Put

$$x \in \mathfrak{A}_k \setminus \mathfrak{A}_{k+1} \rightarrow \varphi_{ik}x = \alpha_{i+1}(xE_i).$$

Then φ_{ik} is a mapping of $\mathfrak{A}_k \setminus \mathfrak{A}_{k+1}$ into $P(\overline{\mathfrak{A}}_i)$. If we assume that $\alpha_{i+1}y = y$ for $y \in \mathfrak{A}_i \setminus \mathfrak{A}_{i+1}$ (as is done, for example, in (2)), then φ_{ii} is simply the embedding of $\mathfrak{A}_i \setminus \mathfrak{A}_{i+1}$ into $P(\overline{\mathfrak{A}}_i)$. We assemble the mappings φ_{ik} ($k \leq i$) into the matrix

$$\Phi = \begin{pmatrix} \varphi_{11} & 0 & \dots & 0 \\ \varphi_{21} & \varphi_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \varphi_{n1} & \varphi_{n2} & \dots & \varphi_{nn} \end{pmatrix}.$$

* That is, a basis of $P(\mathfrak{B})$ consists of all nonzero elements of \mathfrak{B} , which are multiplied in the same way as in \mathfrak{B} , and the zero of $P(\mathfrak{B})$ is identified with the zero of \mathfrak{B} (if \mathfrak{B} has a zero).

It turns out that specifying the factors $\overline{\mathfrak{A}}_i$ ($i = 1, 2, \dots, n$) of the series (1) and the matrix Φ completely determines \mathfrak{A} . Therefore we shall write

$$\mathfrak{A} = (\overline{\mathfrak{A}}_1, \dots, \overline{\mathfrak{A}}_n, \Phi).$$

- Let us show how the question of isomorphism of semigroups from \mathfrak{K} is solved. A necessary condition for isomorphism is equality of the lengths of the ideal composition series of the semigroups. Therefore, considering $\mathfrak{A}, \mathfrak{B} \in \mathfrak{K}$, we may assume:

$$\mathfrak{A} = (\overline{\mathfrak{A}}_1, \dots, \overline{\mathfrak{A}}_n, \Phi), \quad \mathfrak{B} = (\overline{\mathfrak{B}}_1, \dots, \overline{\mathfrak{B}}_n, \Psi).$$

Denote by $L = (\lambda_{ip})$ an $n \times n$ -matrix in each row and each column of which there is exactly one nonzero element. Moreover, if $\lambda_{ip} \neq 0$, then λ_{ip} is an isomorphism of $\overline{\mathfrak{A}}_i$ onto $\overline{\mathfrak{B}}_p$, extended linearly to an isomorphism of $P(\overline{\mathfrak{A}}_i)$ onto $P(\overline{\mathfrak{B}}_p)$. In this case $L\Phi$ and ΨL have a natural meaning.

Theorem 2. For \mathfrak{A} and \mathfrak{B} to be isomorphic it is necessary and sufficient that there exist a matrix L such that

$$L\Phi = \Psi L.$$

4. Let us give a vector interpretation of the semigroup $\mathfrak{A} \in \mathfrak{K}$, $\mathfrak{A} = (\overline{\mathfrak{A}}_1, \dots, \overline{\mathfrak{A}}_n, \Phi)$. Denote by $V(\mathfrak{A})$ the totality of all n -dimensional vectors of the form

$$\bar{x}_i = (0, \dots, 0, x_i, \dots, \varphi_{ki}x_i, \dots),$$

where $x_i \in \mathfrak{A}_i \setminus \mathfrak{A}_{i+1}$, $\{\bar{x}_i\}_p = 0$ for $p \leq i$, $\{\bar{x}_i\}_p = \varphi_{pi}x_i$ for $p \geq i$.^{*} The zero standing in the place of the k -th component x_i is regarded as the zero of $P(\overline{\mathfrak{A}}_k)$. Then

$$\bar{x} \in V(\mathfrak{A}) \rightarrow \{\bar{x}\}_k \in P(\overline{\mathfrak{A}}_k).$$

This makes it possible to define multiplication in $V(\mathfrak{A})$:

$$\bar{x}, \bar{y} \in V(\mathfrak{A}) \rightarrow \{\bar{x} \cdot \bar{y}\}_k = \{\bar{x}\}_k \cdot \{\bar{y}\}_k. \quad (3)$$

Define also a mapping ω of $V(\mathfrak{A})$ onto \mathfrak{A} , putting:

$$\omega\bar{x}_i = x_i. \quad (4)$$

Theorem 3. With respect to the multiplication (3), $V(\mathfrak{A})$ forms a semigroup isomorphic to \mathfrak{A} . An isomorphism of $V(\mathfrak{A})$ onto \mathfrak{A} may be given by the mapping ω .

5. Let us describe the automorphisms of $\mathfrak{A} \in \mathfrak{K}$, $\mathfrak{A} = (\overline{\mathfrak{A}}_1, \dots, \overline{\mathfrak{A}}_n, \Phi)$. For this purpose we replace in item 3 the semigroup \mathfrak{B} by $\mathfrak{A} = (\overline{\mathfrak{A}}_1, \dots, \overline{\mathfrak{A}}_n, \Phi)$. Then the nonzero element λ_{ip} of the matrix L will be an isomorphism of $\overline{\mathfrak{A}}_i$ onto $\overline{\mathfrak{A}}_p$. Define a mapping A of the semigroup \mathfrak{A} , putting for $x \in \mathfrak{A}$

$$Ax = \omega(L \cdot \bar{x}). \quad (5)$$

Here $L \cdot \bar{x}$ is the usual product of a matrix and a vector, and ω is defined by (4).

Theorem 4. The automorphism group of \mathfrak{A} is isomorphic to the multiplicative group \mathfrak{L} of all matrices L such that $L\Phi = \Phi L$. Every automorphism A of the semigroup \mathfrak{A} has the form (5), where $L \in \mathfrak{L}$, and conversely: if $L \in \mathfrak{L}$, then the mapping A , defined by (5), is an automorphism of \mathfrak{A} .

6. Let us call δ a two-sided operator in the semigroup \mathfrak{B} if for all $x \in \mathfrak{B}$ there are defined $\delta x, x\delta \in \mathfrak{B}$ (not necessarily $\delta x = x\delta$). Equality and multiplication of two-sided operators in \mathfrak{B} are defined naturally. By $T\mathfrak{B}$ we denote the semigroup of all such two-sided operators δ in \mathfrak{B} , so that

$$x, y \in \mathfrak{B} \rightarrow \delta(xy) = (\delta x)y, \quad (x\delta)y = x(\delta y), \quad (xy)\delta = x(y\delta).$$

$T\mathfrak{B}$ is nothing other than the semigroup of pairs of linked one-sided translations of \mathfrak{B} ([1], p. 173). If $\mathfrak{B} \in \mathfrak{K}$, then one may regard \mathfrak{B} as a two-sided ideal of $T\mathfrak{B}$. Note the proposition:

Theorem 5. $\mathfrak{A} \in \mathfrak{K} \rightarrow T\mathfrak{A} \in \mathfrak{K}$.

^{*} By $\{x\}_p$ is denoted the p -th component of the vector \bar{x} .

7. We shall call a semigroup \mathfrak{S} an **extension** of a semigroup \mathfrak{X} by means of a semigroup \mathfrak{D} , if \mathfrak{X} is a two-sided ideal of \mathfrak{S} and $\mathfrak{S}/\mathfrak{X} \simeq \mathfrak{D}$. For the construction of semigroups of the class \mathfrak{K} it is enough to be able to construct extensions of any concrete semigroup from \mathfrak{K} by means of a semigroup of the class \mathfrak{C} (this follows from Theorem 1). These extensions are described by

Theorem 6*. Let $\mathfrak{A} \in \mathfrak{K}$, $\mathfrak{B} \in \mathfrak{C}$; θ be the zero of \mathfrak{B} ; $\Omega = \mathfrak{B} \setminus \theta$. As noted in item 6, we may regard \mathfrak{A} as a two-sided ideal of $T\mathfrak{A}$. Suppose further that for a mapping τ one has: a) either τ is a homomorphism of \mathfrak{B} into some semigroup \mathfrak{A}_k such that $\tau\mathfrak{B}$ can be equal to the zero of \mathfrak{A}_k only for $k = n$; b) or τ is a homomorphism of \mathfrak{B} into $T\mathfrak{A}/\mathfrak{A}$, mapping the zero of \mathfrak{B} to the zero of $T\mathfrak{A}/\mathfrak{A}$.

Denoting by α the ideal homomorphism of $T\mathfrak{A}$ generated by \mathfrak{A} , and by α_{k+1} the same as in item 2, put, in case a), $\rho = \alpha_{k+1}^{-1}\tau$, and, in case b), $\rho = \alpha^{-1}\tau$. We now define in $\mathfrak{S} = \mathfrak{A} \cup \mathfrak{C}$ the multiplication “ \circ ” by setting:

$$\begin{aligned} x, y \in \mathfrak{A} &\rightarrow x \circ y = xy; & x, y, xy \in \mathfrak{C} &\rightarrow x \circ y = xy; \\ x, y \in \mathfrak{C}, & \quad xy \notin \mathfrak{C} &\rightarrow x \circ y = \rho x \cdot \rho y & \quad **; \\ x \in \mathfrak{C}, & \quad y \in \mathfrak{A} &\rightarrow x \circ y = \rho x \cdot y, & \quad yx = y \cdot \rho x. \end{aligned} \quad (6)$$

Then \mathfrak{S} , with respect to the multiplication “ \circ ,” is a semigroup—an extension of \mathfrak{A} by means of \mathfrak{B} . Conversely, every extension of \mathfrak{A} by means of \mathfrak{B} can be obtained by the method described in the theorem.

7, A. If \mathfrak{S} is an extension of \mathfrak{A} by means of \mathfrak{B} and the multiplication in \mathfrak{S} is defined by formulas (6), then we shall write $\mathfrak{S} = (\mathfrak{B}, \mathfrak{A}, \rho)$.

7, B. The homomorphisms $\mathfrak{B}, \mathfrak{B} \in \mathfrak{C}$ are described by the method (3). Further, for $\mathfrak{B} \in \mathfrak{C}$ it is easy to construct $T\mathfrak{B}$ ((1), Theorem 6). Therefore we shall have the possibility of an unrestricted application of the construction of Theorem 6 if we show how, knowing $T\mathfrak{B}$ and $T\mathfrak{A}$, one can construct $T\mathfrak{S}$, $\mathfrak{S} = (\mathfrak{B}, \mathfrak{A}, \rho)$. This construction is given by

Theorem 7. Let $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ be the same as in Theorem 6, and let $\mathfrak{S} = (\mathfrak{B}, \mathfrak{A}, \rho)$. Suppose further that $\delta_1 \in T\mathfrak{B}$, $\delta_2 \in T\mathfrak{A}$, and that for $x \in \mathfrak{B}$ the following conditions are satisfied:

$$\begin{aligned} \delta_1 x \in \mathfrak{C} &\rightarrow \delta_2 \cdot \rho x = \rho(\delta_1 x); & x\delta_1 \in \mathfrak{C} &\rightarrow \rho x \cdot \delta_2 = \rho(x\delta_1), \\ \delta_1 x \notin \mathfrak{C} &\rightarrow \delta_2 \cdot \rho x \in \mathfrak{A}; & x\delta_1 \notin \mathfrak{C} &\rightarrow \rho x \cdot \delta_2 \in \mathfrak{A}. \end{aligned}$$

Define in \mathfrak{S} a two-sided operator δ , setting:

$$x \in \mathfrak{A} \rightarrow \delta x = \delta_2 x; \quad x\delta = x\delta_2;$$

$$x, \delta_1 x \in \mathfrak{C} \rightarrow \delta x = \delta_1 x; \quad x, x\delta_1 \in \mathfrak{C} \rightarrow x\delta = x\delta_1;$$

$$x \in \mathfrak{C}, \delta_1 x \notin \mathfrak{C} \rightarrow \delta x = \delta_2 \cdot \rho x; \quad x \in \mathfrak{C}, x\delta_1 \notin \mathfrak{C} \rightarrow x\delta = \rho x \cdot \delta_2.$$

Then $\delta \in T\mathfrak{S}$, and every element δ of $T\mathfrak{S}$ can be obtained by the method described in the theorem.

8. We describe, in conclusion, the homomorphisms of semigroups of the class \mathfrak{K} . It suffices to show how homomorphisms $\mathfrak{A} \in \mathfrak{K}$ are constructed from homomorphisms \mathfrak{A}_i ($i = 1, 2, \dots, n$).

If μ, ν are homomorphisms of the semigroup \mathfrak{S} and $\mu = \lambda\nu$, where λ is an isomorphism of $\nu\mathfrak{S}$, then we shall write $\mu \sim \nu$. The greatest common right divisor ⁽²⁾ of the homomorphisms μ_1, \dots, μ_m of the semigroup \mathfrak{S} will be denoted by o. n. d. μ_i ($i = 1, 2, \dots, m$).

* The construction of Theorem 6 is based on a construction proposed by Clifford ⁽¹⁾.

** $xy \notin \mathfrak{C}$, i.e. $xy = \theta$ in the semigroup \mathfrak{B} .

Theorem 8. Let $\mathfrak{A} \in K$, and let $\bar{\mu}_i$ be a homomorphism of $\bar{\mathfrak{A}}_i$. In the notation of §2, for $x \in \mathfrak{A}$ put

$$\mu_i x = \bar{\mu}_i \alpha_{i+1}(xE_i). \quad (7)$$

Then: a) μ_i is a homomorphism of \mathfrak{A} ; b) if μ is a homomorphism of \mathfrak{A} , then there exist homomorphisms $\bar{\mu}_i$ of the semigroups $\bar{\mathfrak{A}}_i$ ($i = 1, 2, \dots, n$) such that $\mu \sim$ o. n. d. μ_i ($i = 1, 2, \dots, n$), where μ_i is obtained from $\bar{\mu}_i$ by (7).

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