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Abstract

Full Text

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On Equations of Elliptic Type with Discontinuous Coefficients

(Presented by Academician S. L. Sobolev on 21 V 1962)

1. Recently, in a number of works ([1–5]), boundary-value problems for second-order equations of elliptic type with discontinuous coefficients have been studied. In ([1,2]) O. A. Oleinik obtained a solution of the first boundary-value problem as the limit of solutions of the corresponding problems for equations with smooth coefficients. In ([4,5]) V. A. Ilyin, using methods of potential theory and integral equations, proved the solvability of the first and third boundary-value problems.

In the present note, boundary-value problems for equations with discontinuous coefficients are studied by functional methods: on Sobolev classes of functions satisfying certain boundary conditions, an inequality of Gårding type is established, ensuring the existence of a generalized solution. The smoothness of the generalized solution up to the surfaces of discontinuity and the boundary is proved by a method analogous to that used by Nirenberg in ([6]). The same methods are used to study eigenvalue problems. The note considers the case of second-order equations; however, the method can also be extended to equations of higher order.

2. Let

$$\mathcal{L}u = \sum_{j,k=1}^{n} D_j(b_{jk}(x)D_k u) + \sum_{j=1}^{m} p_j(x) D_j u + b(x)u \qquad \left(D_j = \frac{\partial}{\partial x_j};\ b_{jk} = b_{kj}\right) \tag{1}$$

be a second-order differential expression with complex coefficients, defined in a domain G of n-dimensional Euclidean space E_n with boundary Γ. The domain G is divided into two* domains G_1 and G_2 by an $(n-1)$-dimensional surface γ, homeomorphic to a sphere and having no common points with Γ. We shall assume that in G_i, $b_{jk} = b^i_{jk}$, $p_j = p^i_j$, $b = b^i$, and $b^i_{jk}, p^i_j \in C^1(\overline{G}_i)$, $b^i \in C^0(\overline{G}_i)$ $(i = 1, 2)$; Γ is piecewise smooth, and γ is continuously differentiable. In addition, we assume that $\operatorname{Re}\mathcal{L}$ is elliptic in each of the $\overline{G}_i$; $\mathcal{L}^+$ is the formally adjoint expression. Consider the direct sum of Sobolev spaces $W^1_2(G_1) + W^1_2(G_2) = \mathbf{W}^1_2$; $\mathbf{W}^1_2$ is obtained by closing, in the norm

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$$\|u\|_1^2 = \int_G \left\{ |u|^2 + \sum_j |D_j u|^2 \right\} dx,$$

the set of functions defined in G and smooth in each of the $\overline{G}_i$. Similarly, denote $\mathbf{W}_2^2 = W_2^2(G_1) + W_2^2(G_2)$. The symbols $\|\cdot\|_0$, $\|\cdot\|_1$, $\|\cdot\|_2$ will denote the norms, respectively, in the spaces $L_2 = L_2(G)$, $\mathbf{W}_2^1$, $\mathbf{W}_2^2$; $(\ ,\ )$ denotes the scalar product in L_2.

The boundary-value problem is considered in the following formulation. By $\mathbf{W}_2^2(\text{br})$ we denote the set of functions $u \in \mathbf{W}_2^2$ for which

$$m\frac{\partial u}{\partial \mu} + Tu + Qu\Big|_\Gamma = 0; \tag{2}$$

* The case of two domains is considered only for simplicity of formulation. All results are valid for a decomposition into a finite number of domains.

$$a_1 \left.\frac{\partial u}{\partial \psi_1}\right|_\gamma = a_2 \left.\frac{\partial u}{\partial \psi_2}\right|_\gamma; \qquad [u]_\gamma = 0. \tag{3}$$

Here T is a linear differential expression of the first order, defined on functions from $C^1(\Gamma)$ and constituting a linear combination of tangential derivatives with real coefficients from $C^1(\Gamma)$; Q is a linear bounded operator in $L_2(\Gamma)$; m is a constant, equal to zero when $T = 0$, $Q = 1$, and equal to 1 in the remaining cases; a_1, a_2 are positive functions, $a_i \in C^2(\overline{G_i})$; $\frac{\partial}{\partial \psi_i}$ is differentiation in the conormal direction,

$$\frac{\partial}{\partial \psi_i} = \sum_{j,k} b_{jk}^i \nu_k^i D_j u$$

(ν_k^i are the components of the normal to γ exterior with respect to G_i); $[u]_\gamma = u|_{\gamma-0} - u|_{\gamma+0}$, the symbols $\gamma - 0$, $\gamma + 0$ mean that limiting values are taken from different sides of γ. From the embedding theorems (7) it follows that $\mathbf{W}_2^2(\text{gr})$ is a subspace of $\mathbf{W}_2^2$. In the case $T \neq 0$, twice continuous differentiability is required on the boundary Γ. We shall assume T and Q to be such that $u \in \mathbf{W}_2^2(\text{gr})$, considered only on Γ, form a dense set in $L_2(\Gamma)$.

Definition 1. A function $u \in \mathbf{W}_2^2(\text{gr})$ for which $\mathcal{L}u = f$ will be called a **smooth solution** of the boundary-value problem

$$\mathcal{L}u = f, \qquad u \in (\text{gr}). \tag{4}$$

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In particular, by choosing different m, T, and Q, we obtain the first, second, and third boundary-value problems and the problem with an oblique derivative.

Denote by $\mathbf{W}_2^2(\mathrm{gr})^+$ the set of those $v \in \mathbf{W}_2^2$ for which $(\mathcal{L}u, av) = (u, \mathcal{L}^+(av))$, $u \in \mathbf{W}_2^2(\mathrm{gr})$; here $a = a_i$ in G_i. Applying Green' s formula to the domains G_1 and G_2 and taking into account (2), (3), we obtain that $\mathbf{W}_2^2(\mathrm{gr})^+$ consists of those and only those $v \in \mathbf{W}_2^2$ for which

$$m\frac{\partial(av)}{\partial\mu} + T^+(av) + Q^*(av) - m\beta av\big|_\Gamma = 0; \tag{5}$$

$$\left.\frac{\partial(a_1\overline{v})}{\partial\mu_1} - \frac{\partial(a_2\overline{v})}{\partial\mu_2} - (a_1\beta_1 - a_2\beta_2)\overline{v}\right|_\gamma = 0, \qquad [v]_\gamma = 0, \tag{6}$$

where

$$\frac{\partial}{\partial\mu} = \sum_{j,k}\overline{b}_{jk}\nu_k D_j; \qquad \beta_i = \sum_j p_j^i \nu_j^i.$$

T^+ is the expression formally adjoint to T; Q^* is the operator adjoint to Q in $L_2(\Gamma)$; $\mathbf{W}_2^2(\mathrm{gr})^+$ is also a subspace of $\mathbf{W}_2^2$, and $\mathbf{W}_2^2(\mathrm{gr})^{++} = \mathbf{W}_2^2(\mathrm{gr})$.

If u is a smooth solution of problem (4), then $(u, \mathcal{L}^+(av)) = (f, av)$, $v \in \mathbf{W}_2^2(\mathrm{gr})^+$. With the aid of integration by parts we transfer one differentiation on u to the left-hand side of this equality. We obtain

$$\begin{aligned} B(u, av) \equiv\ & -\sum_{j,k}(b_{jk}D_j u, D_k(av)) + \sum_i (p_{jD} ju, av) + (bu, av) \\ & - m\int_\Gamma u(\overline{T^*(av) + Q^*(av)})\, dx = (f, av). \end{aligned} \tag{7}$$

Denote by ${\mathbf{W}'_2}^1$ the closure of $\mathbf{W}_2^2(\mathrm{gr})^+$ in the metric of $\mathbf{W}_2^1$. The bilinear functional $B(u, av)$ may, by continuity, be extended to all of ${\mathbf{W}'_2}^1$. The set ${\mathbf{W}'_2}^1$, and likewise the set ${\mathbf{W}''_2}^1$ of elements of the form av ($v \in {\mathbf{W}'_2}^1$), are subspaces of $\mathbf{W}_2^1$, dense in L_2.

Definition 2. A function $u \in {\mathbf{W}'_2}^1$ will be called a **generalized solution** of problem (4) if

$$B(u, av) = (f, av), \qquad v \in {\mathbf{W}'_2}^1. \tag{8}$$

3. By means of integration by parts, using the ellipticity condition, one can prove the following analogue of Gårding' s inequality.

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Lemma. *Let the coefficients $p_j(x)$ be real. There exist constants $K \geqslant 0$, $C > 0$, independent of $v \in \mathbf{W}_2^2(\mathrm{gr})^+$, such that*

$$\operatorname{Re}(\mathcal{L}^+(av) + Kv, v) > C\|v\|_1^2, \qquad v \in \mathbf{W}_2^2(\mathrm{gr})^+. \tag{9}$$

Thus, if $\operatorname{Re} b(x)$ is sufficiently positive, we obtain

$$|B(v, av)| > C\|v\|_1^2, \qquad v \in \mathbf{W}_2'^1. \tag{10}$$

On the positive space $H^+ = \mathbf{W}_2'^1$ and the zero space $H^0 = L_2(G)$ we construct the space with negative norm $H^- = \mathbf{W}_2'^{-1}$ (see ([8],[9])).

Theorem 1. *Let $p_j(x)$ be real, and let $\operatorname{Re} b(x)$ be so positive that inequality (10) holds. Then for every $f \in \mathbf{W}_2'^{-1}$ there exists one and only one generalized solution $u \in \mathbf{W}_2'^1$ of problem (4).*

The proof follows directly from inequalities (10) and $|B(u, av)| \leqslant C_1\|u\|_1\|v\|_1$ $(u, v \in \mathbf{W}_2'^1)$ with the aid of the Vishik–Lax–Milgram lemma (see, for example, ([6])).

Theorem 2. *Suppose that all the assumptions of Theorem 1 hold. In addition, assume the following: the boundaries Γ and γ are twice continuously differentiable; b_{jk}^i, p_j^i, $b^i \in C^1(\overline{G}_i)$, $f \in L_2(G)$; the coefficients of the expression T belong to $C^2(\Gamma)$, and Q is the operator of multiplication by $\sigma(x) \in C^2(\Gamma)$. Then the generalized solution $u \in \mathbf{W}_2'^1$ of problem (4) belongs to $\mathbf{W}_2^2$ and, consequently, is a smooth solution of this problem.*

The proof of the theorem is carried out by the method used by Nirenberg in ([6]). It is enough to prove that for every point $x_0 \in \overline{G}$ there exists a neighborhood $V = V(x_0)$ such that in it $u \in W_2^2(V \cap G_1) + W_2^2(V \cap G_2)$. We restrict ourselves to the case when $x_0 \in \gamma$. In this case one may assume that γ near x_0 is a piece of an $(n-1)$-dimensional plane (this can be achieved by means of a suitable twice continuously differentiable homeomorphism E_n). Let $U \subset G$ be a neighborhood of x_0 in E_n, whose intersection with γ lies on the flat piece γ. Construct an auxiliary function $\xi(x) \in C^\infty(G)$ such that $\xi = 0$ outside U, $\xi \equiv 1$ in some neighborhood V of the point x_0 contained in U in E_n, $0 \leqslant \xi \leqslant 1$. Let the equation of the flat piece γ under consideration be $x_n = 0$. If $x = (x_1, \dots, x_n) \in \overline{G}_i \cap U$ and h is sufficiently small, then the point $x_h^k = (x_1, \dots, x_k + h, \dots, x_n)$ $(k = 1, \dots, n-1)$ lies in $\overline{G}_i$ $(i = 1, 2)$. For any function $g(x)$ denote

$$g_k^h = \frac{1}{h}(g(x_h^k) - g(x))$$

and regard g_k^h as a function of x. In the proof the inequality used is

$$|B((\xi u)_k^h, av)| \leqslant |B(u, a\xi v_k^{-h})| + C\|v\|_1 \qquad (k = 1, \dots, n-1), \tag{11}$$

where $u \in \mathbf{W}_2^1$, and the constant C is independent of $v \in \mathbf{W}_2^1$. Let u be a generalized solution of problem (4), where $f \in L_2(G)$. Then from equality (8)

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it follows that

$$|B(u,av)| \leqslant C\|v\|_0 \qquad (v \in \mathbf{W}_2'^1).$$

Substituting in this inequality ξv_k^{-h} instead of v* and using inequalities (11) and (10), we obtain $\|(\xi u)_k^h\|_1 \leqslant C_1$. Since $\xi \equiv 1$ in some neighborhood V of the point x_0, in this neighborhood $\|u_k^h\|_1 \leqslant C_1$. Passing to the limit as $h \to 0$, we see that $\|D_k u\|_1 \leqslant C_1$ $(k = 1, \dots, n-1)$, i.e.

$$D_{jD}ku \in L_2(V) \qquad (j = 1, \dots, n;\ k = 1, \dots, n-1). \tag{12}$$

* It is not hard to show that $\mathbf{W}_2'^1$ consists of functions which near γ belong to $\mathbf{W}_2^1$ and are continuous.

It remains to prove that $D_{nl}^2 u \in L_2(V)$. We obtain this by expressing $D_{nl}^2 u$ from the equation $\mathcal{L}u = f$ in terms of f, u, the first derivatives of u, and the second derivatives of the form (12).

4. The preceding considerations make it possible to establish the following theorem.

Theorem 3. *Let the assumptions of Theorem 2 be satisfied. Consider the mapping* $\Lambda : u \to \mathcal{L}u$ $(u \in \mathring{W}_2^2(\Gamma),\ \mathcal{L}u \in L_2)$ *as an operator acting in one of the following pairs of spaces:*

$$\mathring{W}_2^2(\Gamma) \to L_2, \qquad \mathring{W}_2^1 \to \mathring{W}_2^{-1}, \qquad L_2 \to \mathring{W}_2^{-2},$$

where $\mathring{W}_2^{-2}$ *is the negative space constructed from the null space* L_2 *and the positive space* $\mathring{W}_2^2 = \{av;\ v \in \mathring{W}_2^2(\Gamma)^+\}$. *Then the closure* $\overline{\Lambda}$ *of this operator is a homeomorphism between the spaces of the corresponding pairs (for the first pair the homeomorphism is already* Λ*).*

Remark. In the case of the second pair of spaces, the required smoothness of the coefficients and of the boundary can be lowered.

5. Let us also consider the eigenvalue problem; we shall investigate the generalized and classical solvability of the problem

$$\mathcal{L}u - \lambda u = f \in L_2, \qquad u \in (\Gamma), \tag{13}$$

where λ is a complex parameter.

Theorem 4. *Let all the assumptions of Theorem 1 be satisfied, except for the requirement of sufficient positivity of* $\operatorname{Re} b(x)$. *Then normal solvability holds for problem (13), i.e. it is Fredholm. The generalized eigenfunctions belong to* $\mathring{W}_2^1$. *If the smoothness requirements of Theorem 2 are fulfilled, then the generalized eigenfunctions are smooth, i.e. belong to* $\mathring{W}_2^2$.

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his book on boundary-value problems and expansions in eigenfunctions, which helped the authors in solving the problems considered.

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