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Abstract

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ON THE DISTRIBUTION OF ANALOGUES OF PRIME NUMBERS

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For a semigroup (with respect to multiplication) G of real numbers $a \geqslant 1$ with the given asymptotic

$$\sum_{a \leqslant x} 1 \sim \alpha x \quad (x \to \infty),$$

Beurling (^{1}) proved an asymptotic distribution law for the basic numbers $b \in G$. Forman and Shapiro (^{2}) divided G into classes H_i $(1 \leqslant i \leqslant h)$, forming a group Γ (for any $a \in H_i,\ a' \in H_j$ the product aa' belongs to the class H_k, whose index depends only on i and j, but not on a, a'), and moreover

$$\sum_{a \leqslant x,\ a \in H_i} 1 = \alpha_i x + O\left(x^{1-\vartheta}\right), \tag{1}$$

where $\alpha_i > 0,\ \vartheta \in (0,1]$ are constants. They proved that under such conditions almost all basic numbers b are distributed asymptotically uniformly among the classes H belonging to some subgroup Γ_0 of the group Γ; in the remaining classes the number of basic numbers $b \leqslant x$ is a quantity of lower order. In the work of Forman and Shapiro the number of classes is bounded $(h \leqslant 1)$.

The purpose of the present note is to set forth a proof scheme for a general theorem on the existence of a basic number not exceeding a certain bound in any class H. In this case it is necessary to assume that in (1) α_i does not depend on i, and to abandon the restriction $h \leqslant 1$. Accordingly, an unbounded parameter $D \geqslant 2$ is introduced, it is assumed that the number of classes $h \leqslant D$, and (1) is replaced by the law

$$\sum_{a \leqslant x,\ a \in H_i} 1 = \alpha x + O\left(D^{c_1} x^{1-\vartheta}\right), \qquad \alpha = D^l, \tag{2}$$

where l, c_1, and ϑ do not depend on i $(0 \leqslant c_1 \leqslant 1,\ 0 < \vartheta \leqslant 1,\ 0 \geqslant l \leqslant 1)$. If h is an even number, then, denoting by Γ_j any subgroup of index 2 of the group of classes H, by virtue of (2) we have

$$\sum_{a\leqslant x,\ a\in\Gamma_j}\frac{1}{a}=\frac{1}{2}\alpha h\log x+A_j+O\left(hD^{c_1}x^{-\vartheta}\right),$$

$$\sum_{a\leqslant x,\ a\notin\Gamma_j}\frac{1}{a}=\frac{1}{2}\alpha h\log x-B_j+O\left(hD^{c_1}x^{-\vartheta}\right),\tag{3}$$

where $A_j=A_j(D),\ B_j=B_j(D)$ do not depend on x. Under these conditions the following theorem holds:

Theorem. I. If $\vartheta>1/2$, then there exists a positive constant c (depending only on l,c_1,ϑ) such that for any class H_i and for any $x\geqslant 1$ there is a basic number $b\in H_i$ in the interval (x,xD^c). If h is an odd number, then the same conclusion is true also for $\vartheta\leqslant 1/2$.

II. Let h be an even number and $\vartheta\leqslant 1/2$. If there exists a constant c_2 such that in (3), for all j,

$$A_j-B_j>D^{-c_2},\tag{4}$$

then the conclusion of the first part of the theorem holds, but the constant depends on ϑ,l,c_1,c_2.

III. If h is an even number, $h\alpha=D^{o(1)}$ $(D\to\infty)$, and, independently of $\vartheta>1/2$ or $\vartheta\leqslant 1/2$, (4) holds for any $c_2>0$, then a basic number $b\in H_i$ is already found in the interval (x,xD^{ε}) for any positive $\varepsilon\leqslant c$,

$D>D_0(\varepsilon)$ and $x\geq D^{c'\log(c/\varepsilon)}$, where the constant $c'=c'(l,c_1,\vartheta)$ does not depend on ε. For odd h the same conclusion is always valid.

The theorem is of interest only for $x<\exp(D^c)$. In application to an arithmetic progression I (with $x=1$) gives the estimate $\min p<D^c$ of Yu. V. Linnik (^{3}) for the least prime number $p\equiv l \pmod D$, and III reduces to the result proved by me earlier (^{4}).

The numbers a represented by all inequivalent binary quadratic forms $\psi_1,\dots,\psi_h$ of fixed discriminant d form a semigroup and are distributed asymptotically equally among the classes (if each a is counted with the required number of repetitions; cf. (^{5}), Theorems 204, 208). By Landau' s method (^{6}) one can prove the estimate $\pi D^{-1/2}x+O(D^{3/2}x^{1/3})$ for the number of integral points of the ellipse $\psi(u,v)\leq x$ with discriminant $-D$, which permits the application of I. By a deeper arithmetic (briefly described in (^{7},^{8})) one can show that for quadratic forms $\psi(u,v)$ of any discriminant the conditions III hold.

For the sequence of norms of ideals distributed by classes $\bmod\ \mathfrak{f}$ in any algebraic field of degree n and discriminant Δ, by contour integration of the function $D^{2ns}\Gamma(s)L(s,\chi)L(s,\chi_0)$ (where $D=|\Delta|N\mathfrak{f}$ and $L(s,\chi)$, $L(s,\chi_0)$ are Hecke L-functions with a real, respectively principal, character $\bmod\ \mathfrak{f}$) one can prove (4) with the constant $c_2 = 2n$ (cf. (^{9}), pp. 104-105), which permits the application of II. The corresponding result has been proved by another method in (^{9}).

The theorem of the present note shows that the known estimate of the least prime number in arithmetic and other important progressions can be proved without using the functional equation and the existence of the corresponding L-series in the half-plane $\sigma < 1 - \vartheta$ of the complex variable $s = \sigma + it$.

Let, further, $c_3, c_4, \ldots$ denote positive constants depending only on ϑ, l, c_1, c_2; the constants in $\ll$ are of the same kind.

In consequence of (2), the function

$$\zeta(s, H) = \sum_{a \in H} a^{-s} \qquad (\sigma > 1)$$

exists in $\sigma > 1 - \vartheta$ (with the exception of a simple pole $s = 1$ with residue α) and satisfies the estimate $\zeta(s, H) \ll D^{c_3}(|s-1|^{-1} + |t|)$. Let $\chi(H)$ denote the characters of the classes H and let $\chi(a) = \chi(H)$ for all $a \in H$. Then the function

$$\zeta(s, \chi) = \sum_H \chi(H)\zeta(s, H) = \sum_a \chi(a) a^{-s} = \prod_b (1 - \chi(b) b^{-s})^{-1} \qquad (\sigma > 1) \quad (5)$$

is regular in the half-plane $\sigma > 1 - \vartheta$, except for a simple pole $s = 1$ (with residue $h\alpha$) for the function $\zeta(s, \chi_0)$ with the principal character χ_0. By known methods (cf. (^{9}), p. 95) it is proved that in $\sigma > 1 - \frac{1}{4}\vartheta$ we have

$$\frac{\zeta'}{\zeta}(s, \chi) - \sum_{|s-\rho| < \vartheta/2} \frac{1}{s - \rho} + \frac{e_0}{s - 1} \ll \log D(1 + |t|), \qquad (6)$$

where ρ runs over the zeros of $\zeta(s, \chi)$, $e_0 = 1$ for the principal character and $e_0 = 0$ for the remaining χ. Let $\nu = \nu(r; \chi, t)$ denote the number of zeros of $\zeta(s, \chi)$ in $|w - 1 - it| \leq r$. Then, for $e_1 / \log D(1 + |t|) \leq r \leq \frac{1}{4}\vartheta$ (where e_1 is any positive constant ≤ 1), we have

$$\nu \leq c_4 r \log D(1 + |t|), \qquad c_4 = c_4(e_1). \qquad (7)$$

By means of these estimates the following is proved.

Lemma 1. For a suitable $c_5 > 0$, in the region $E(\sigma \geq 1 - c_5 / \log D(1 + |t|))$ the function $\zeta(s, \chi)$ with complex χ has no zeros. For no more than one real character $\chi = \chi'$ is a simple real zero $\rho' \leq 1$ possible in E.

If $\vartheta > 1/2$, then, by contour integration of the function $D^{c_6}\Gamma(s)\zeta(s,\chi')\zeta(s,\chi_0)$ (in the case $\chi' \neq \chi_0$), it is proved that $1-\rho' > D^{-c_7}$. For $\vartheta \leqslant 1/2$ the same estimate follows from (4).

By Turán' s method $^{(10)}$ the following fundamental result is proved.

Lemma 2. *Let* k, A, D, λ, τ_0 *denote unrestricted parameters, with* k *an integer* $\geqslant 2$, $A > 0$, $D \geqslant 2$, $0 \leqslant c_0 \leqslant \lambda \leqslant \vartheta_0 \log D$ $(0 < \vartheta_0 \leqslant 1)$, $-D \leqslant \tau_0 \leqslant D$. *Let, in the half-plane* $\sigma > 1-\vartheta_0$, $F(s)$ *be a meromorphic function with simple poles* ρ *to the left of (or on) the line* $\sigma = 1$, *and, denoting by* m_ρ *the residue of* $F(s)$ *at the point* $s = \rho$, *suppose that for all* t *we have*

$$\sum_{|\rho-1-it|\leqslant r} |m_\rho| \leqslant C_0 r \log D(1+|t|) \quad (c_0/\log D(1+|t|) \leqslant r \leqslant \vartheta_0),$$

where C_0 *(and below* $C_1 > 1$*) depends only on* c_0, ϑ_0. *Further suppose that, for all* ρ *in* $(\sigma > 1-\vartheta_0,\ |t-\tau_0| < 1)$, *the* m_ρ *are positive integers. Let, for real* τ,

$$I(\tau,k,A) = -\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\left(\frac{e^{3As}-e^{As}}{2As}\right)^k F(1+s+i\tau)\,ds. \tag{8}$$

If under these conditions in the square Q $(1-\lambda/\log D \leqslant \sigma \leqslant 1,\ |t-\tau_0| \leqslant \lambda/2\log D)$ *there is at least one point* ρ, *then for suitable* C_1 *and for any* $C \geqslant C_1$ *there exists* $k \geqslant \max(2, C\lambda)$, $k < (C+C_1)\lambda$, *such that at all points of the interval* $|\tau-\tau_0| \leqslant \lambda/2\log D$ *we have*

$$|I(\tau,k,\lambda^{-1}\log D)| > \exp(-C_2\lambda), \qquad C_2 = C_2(C_0,C). \tag{9}$$

It should be noted that if $F(s) = \sum_a c_a a^{-s}$ $(\sigma > 1)$, then (cf. $^{(10)}$, Lemma I)

$$I(\tau,k,A) = \sum_a c_a R(a) a^{-1-i\tau}, \quad \text{where} \quad R(a) \begin{cases} < 3/A\sqrt{k} \text{ for } a \in (e^{kA}, e^{3kA}), \\ = 0 \text{ for the remaining } a. \end{cases} \tag{10}$$

By Selberg' s method $^{(11)}$ it is proved that, as a consequence of (2),

$$\sum_{b^n \leqslant x,\ b^n \in H_i,\ n \geqslant 1} \log b \ll x/h, \qquad \text{if } x > D^{c_8}. \tag{11}$$

Let $I_\chi(\tau,k,A)$ denote the transform (8) of the function $F(s) = \zeta'/\zeta(s,\chi)$. Using (11), one proves the estimate

$$\sum_\chi |I_\chi(\tau_0,k,A)|^2 < c_9 \qquad (\text{for } kA > c_8)$$

(cf. [10], Lemma II), from which, together with (9) (with $\tau = \tau_0$), it follows that the number of functions $\zeta(s,\chi)$, each of which has at least one zero $\rho \in Q$, does not exceed $\exp(c_{10}\lambda)$. Hence from (7) it follows:

Lemma 3. *The number of zeros of the function* $\prod_\chi \zeta(s,\chi)$ *belonging to the rectangle*

$$(1-\lambda/\log D \leqslant \sigma \leqslant 1,\ |t| \leqslant e^\lambda/\log D) \quad (c_5 \leqslant \lambda \leqslant \tfrac{1}{4}\vartheta \log D)$$

does not exceed $\exp(c_{11}\lambda)$.

Next we assume that there exists an exceptional zero $\rho' = 1-\delta$ of the function $\zeta(s,\chi')$ with real character χ', and that $\rho = 1-\lambda/\log D + i\gamma$ ($|\gamma| \leqslant D,\ c_5 \leqslant \lambda \leqslant \frac{1}{4}\vartheta\log D$) is a zero $\neq \rho'$ of some function $\zeta(s,\chi)$. If $\chi \neq \chi_0 \neq \chi'$, then, denoting by I_χ the transform (8) of the function

$$F(s) = \zeta'/\zeta(s,\chi)+\zeta'/\zeta(s+\delta,\chi\chi') = -\sum_{b,\ n\geqslant 1} b^{-ns}\chi(b^n)(1+\chi'(b^n)b^{-n\delta})\log b \quad (\sigma>1),$$

as a consequence of (9), (10) we have

$$\sum_{D^B<b<D^{3B}} \frac{\log b}{b}(1+\chi'(b)b^{-\delta}) > \exp(-c_{12}\lambda)\log D$$

$$(c_8 < B \leqslant 1).$$

One may assume that $1-b^{-\delta} < \exp(-2c_{12}\lambda)$ for all

$b \ll D^{3B}$ (cf. [12], p. 347). Then we obtain the inequality

$$\sum_{\substack{D^B<b<D^{3B}\\ \chi'(b)=1}} \frac{1}{b} > \exp(-c_{13}\lambda). \tag{12}$$

Denoting

$$\zeta(s,\chi')\,\zeta(s,\chi_0) = \sum_a g(a)a^{-s} \quad (\sigma>1), \qquad \mu = \zeta(1,\chi')\operatorname*{Res}_{s=1}\zeta(s,\chi_0),$$

we have

$$\sum_{a<D^B}\frac{g(a)}{a}\sum_{D^B<b<D^{3B}}\frac{1}{b} \ll \sum_{D^B<a<D^{4B}}\frac{g(a)}{a} <$$

$$< 8 \sum_{a > D^B} \frac{g(a)}{a} \{\exp(-D^{-4B}a) - \exp(-D^{-B}a)\} < 32B\mu \log D.$$

The last inequality is proved by contour integration of the function $D^{\alpha(s-1)}\Gamma(s-1)\zeta(s,\chi')\zeta(s,\chi_0)$ with $\alpha = 4B$ and $\alpha = B$ (cf. (^{9}), pp. 105, 139-140); here it is assumed that B is sufficiently large, and if $\vartheta \ll 1/2$, then (4) is applied. Hence from (12)

$$\mu > \frac{\exp(-c_{14}\lambda)}{\log D} \sum_{a < D^B} \frac{g(a)}{a}.$$

By contour integration of the function $D^{\frac{1}{2}C(s-\rho')}\Gamma(s-\rho')\zeta(s,\chi')\zeta(s,\chi_0)$ one proves the inequality

$$c_{15} > \exp(-c_{14}\lambda)/\delta \log D \tag{13}$$

(cf. (^{12}), p. 350). In a similar way the same inequality is proved in the cases $\chi = \chi_0 \neq \chi'$, $\chi \neq \chi_0 = \chi'$, $\chi = \chi^0 = \chi'$ (cf. (^{9}), pp. 136-146). A simple consequence of (13) is (cf. (^{9}), pp. 146-147)

Lemma 4. For a suitable $c_{16} > 0$ and

$$\lambda_0 = c_{16} \log \frac{e^{c_{16}}}{\delta_0 \log D}, \qquad \text{where } \delta_0 = \begin{cases} \delta = 1 - \rho', & \text{if } \delta \ll c_{16}/\log D, \\ c_{16}/\log D & \text{otherwise,} \end{cases}$$

the rectangle $(1 - \lambda_0/\log D \ll \sigma \ll 1,\ |t| \ll D)$ contains no zeros $\rho \neq \rho'$ of the function

$$\prod_{\chi} \zeta(s,\chi).$$

On the basis of Lemmas 1, 3 and 4 the theorem is proved by a variant of Rodosskii’ s method ((^{12}), Ch. III; (^{9}), pp. 260-263 and (^{4})), which begins with contour integration of the function $x^{s-\sigma_1}\exp\{y(s-\sigma_1)^2\}\zeta'/\zeta(s,\chi)$ (where $\sigma_1 = 1 - \frac{1}{8}\vartheta$, $y > 1$), the contour being shifted from $\sigma = 2$ to a broken line (differing little from $\sigma = \sigma_1$), from which the zeros of $\zeta(s,\chi)$ are at a known distance (cf. (^{9}), pp. 267-268).

Although all the mentioned corollaries of the theorem were proved by a variant of Rodosskii’ s method, which depends essentially on the distribution of small prime numbers (cf. (^{9}), pp. 133 and 238, where the sum $\sum_{p<Z} p^{-1} \log p$ must not be too large and at the same time Z must not be too small), the theorem

itself cannot be proved by this method (i.e. without applying Lemma 2), since (2) gives almost no information about the distribution of small a.

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Note: Figure translations are in progress. See original paper for figures.

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