V. G. ZHITOMIRSKII

1962

SovietRxiv

Abstract

Full Text

V. G. ZHITOMIRSKII

TRIANGULAR GROUPS OF AUTOMORPHISMS OF QUASI-OPERATOR GROUPS

(Presented by Academician A. I. Mal' cev, 15 I 1962)

Let G be a group and Ω some ring of principal ideals. Suppose that to each element $g \in G$ and to each $\omega \in \Omega$ there corresponds uniquely an element of the group G (we shall denote it by g^ω), and that the following relations hold:

1. $g^{\omega+\nu} = g^\omega g^\nu$ for any $g \in G$ and $\omega, \nu \in \Omega$.
2. $g^{\omega\nu} = (g^\omega)^\nu$ for any $g \in G$ and $\omega, \nu \in \Omega$.
3. For the identity 1 of the ring Ω, $g^1 = g$ for all $g \in G$.
4. For the identity element e of the group G and any $\omega \in \Omega$, $e^\omega = e$.
5. $x^{-1}g^\omega x = (x^{-1}gx)^\omega$ for any $x, g \in G$ and $\omega \in \Omega$.

A subgroup $H \subset G$, as usual, is called Ω-admissible if, for any $h \in H$ and $\omega \in \Omega$, $h^\omega \in H$. The intersection of any set of Ω-admissible subgroups is Ω-admissible. Therefore it makes sense to speak of the Ω-closure of a set of elements $M \subset G$. The following relation connects the operation in G with the action of the elements of Ω.

6. For any $a, b \in G$ and $\omega \in \Omega$,

$$(ab)^\omega = a^\omega b^\omega c,$$

where c belongs to the Ω-closure of the commutant of the subgroup generated by the elements a and b:

$$c \in (K\{a, b\})^\Omega.$$

If relations 1–6 hold, then we shall call G an Ω-group, and the elements of Ω quasi-operators. Let us note that a similar axiomatics, for other purposes, was considered by N. F. Sesekin ([6]).

We shall call an Ω-group G a group without Ω-torsion if from $g^\omega = e$ it follows that $g = e$ for any nonzero element $\omega \in \Omega$. A group without Ω-torsion will be called an ΩR-group if, for any (nonzero) $\omega \in \Omega$, from $a^\omega = b^\omega$ it follows that $a = b$. An Ω-admissible subgroup H of an ΩR-group G will be called Ω-isolated if from $g^\omega \in H$ it follows that $g \in H$. The intersection of any set of Ω-isolated subgroups is Ω-isolated. As usual, the Ω-isolator $I(M)$ of a set M of elements of an ΩR-group G will mean the intersection of all Ω-isolated subgroups containing the set M.

The concept of an ΩR-group is a generalization of the concept of an R-group introduced by P. G. Kontorovich ([1]). Ordinary R-groups are ΩR-groups with the ring of integers as Ω. A vector space over a field of characteristic zero is a commutative ΩR-group. An example of a noncommutative ΩR-group is the well-known group of real numbers with multiplication law

$$(x_1, y_1)(x_2, y_2) = (x_1 + x_2 e^{-y_1},\, y_1 + y_2),$$

if, for any real ω, one sets

$$(x, y)^\omega = \left(x \frac{1 - e^{-\omega y}}{1 - e^{-y}},\, \omega y \right).$$

Many facts from the theory of R-groups and, in particular, from the theory of locally nilpotent groups without torsion ([1–4]), carry over to ΩR-groups. We shall give some of them.

In an ΩR-group, the centralizer of any set of elements and, consequently, the center of an ΩR-group is an Ω-isolated subgroup.

A group without Ω-torsion is an ΩR-group if and only if its factor group by its center is an ΩR-group.

The set of elements of the form g^ω, for fixed g and all possible $\omega \in \Omega$, is an (Ω-admissible) subgroup, which we shall call Ω-cyclic. If, in an (Ω-admissible) subgroup, every finite set of elements belongs to some Ω-cyclic subgroup, then such a subgroup is naturally called Ω-locally cyclic.

In an ΩR-group the Ω-isolator $I(g)$ of any element g is an Ω-locally cyclic group, and the Ω-isolators of distinct elements either do not intersect or coincide.

Every locally nilpotent Ω-group without Ω-torsion is an ΩR-group.

In a nilpotent ΩR-group the normalizer of an Ω-isolated subgroup is Ω-isolated.

If G is a locally nilpotent ΩR-group, then every element a of the Ω-isolator $I(H)$ of an arbitrary Ω-admissible subgroup H has the property that, for some $\omega \in \Omega$, $a^\omega \in H$.

If G is an ΩR-group and H is its subgroup possessing an ascending central series, then the Ω-isolator $I(H)$ also possesses an ascending central series, and the classes of H and $I(H)$ are one and the same.

In an ΩR-group the Ω-isolator of a locally nilpotent subgroup is locally nilpotent.

The locally nilpotent radical $R(G)$ of an ΩR-group G is Ω-isolated.

The proofs of these facts are, in the main, analogous to the proofs of the corresponding facts in the theory of R-groups. The principal changes are connected with the necessity of checking the Ω-admissibility of the subgroups under consideration and with considering the properties of Ω-closures.

An automorphism φ of an Ω-group G will be called an Ω-automorphism if, for all $g \in G$ and $\omega \in \Omega$, $\varphi(g^\omega) = \varphi(g)^\omega$. A group Φ of Ω-automorphisms of an ΩR-group G will be called scalar if the group G is covered by Φ-admissible Ω-locally cyclic subgroups.

A scalar group of automorphisms is abelian.

If in an ΩR-group G there is a normal series consisting of Φ-admissible Ω-isolated subgroups such that in every factor of this series Φ induces a scalar group of automorphisms, then Φ will be called a triangular group of automorphisms. Together with the group of Ω-automorphisms Φ we shall consider the group $\overline{\Phi}$, equal to the product of Φ and the group $\hat{G}$ of inner automorphisms of the group G.

Theorem. Let G be an ΩR-group with the maximality condition for Ω-isolated subgroups, and let Φ be a group of its Ω-automorphisms. Suppose also that $\overline{\Phi}$ is triangular relative to $R(G)$ and Φ is triangular relative to the factor group $G/R(G)$.

Then:

1. The factor group $\Phi/R(\Phi, G)$ of the group Φ by the locally stable radical $R(\Phi, G)$ (5) is abelian.

2. $R(\Phi, G)$ is a nilpotent group.

3. In $R(\Phi, G)$ there is a series invariant under Φ, the factors of which are isomorphic to subgroups of the factors of a triangular series in $R(G)$.

In proving the theorem the following proposition is used essentially.

Let G be an ΩR-group and let H be its normal divisor that is a maximal Ω-isolated subgroup such that the group Ψ of Ω-automorphisms of the group G induces the identity automorphism in H and in the factor group G/H. Suppose, moreover, that in H there is given a series invariant in G with Ω-locally cyclic factors

$$E = H_0 \subset H_1 \subset \cdots \subset H_i \subset \cdots \subset H_k \subset H.$$

Form in Ψ the series

$$E = \Psi_0 \subset \Psi_1 \subset \cdots \subset \Psi_i \subset \cdots \subset \Psi_k \subset \Psi,$$

where Ψ_i is the Ψ-centralizer of the factor group G/H_i.

Then:

1. The series in Ψ is invariant under every containing group of automorphisms Ψ, relative to which the series in H is admissible.

2. The factors of the series in Ψ are isomorphic to subgroups of the factors of the series in H.

We shall present the main stages of the proof of the theorem. The following is proved. A $\overline{\Phi}$-triangular series in $R(G)$ is finite, has Ω-locally cyclic factors, and is invariant under $\hat{G}$. A Φ-triangular series in $G/R(G)$ is finite. Therefore

the triangular series in the group G is finite. The locally stable radical $R(\Phi, G)$ coincides with the Φ-centralizer of the triangular series in G and, consequently, is nilpotent. The factor group $\Phi/R(\Phi, G)$, being a subdirect product of abelian groups, is itself abelian. The series in $R(\Phi, G)$ indicated in the theorem is constructed on the basis of the proposition given above, if one takes into account that $R(\Phi, G)$ induces the identity automorphism in $G/R(G)$ and that in the group G the maximality condition holds for Ω-isolated subgroups.

Let us apply the results obtained to the study of groups preserving the order of automorphisms of an ordered Ω-group.

Consider an Ω-group G, for which Ω is an ordered field. In the case when Ω is a field, every Ω-group is, obviously, an ΩR-group, and in it the notions of Ω-admissibility and Ω-isolation coincide. We shall call such a group G an ordered Ω-group if G is an ordered group and 1) from $a > b$ it follows that $a^\omega > b^\omega$ for every positive element of the field; 2) from $a > e$ and $\omega_1 > \omega_2$ it follows that $a^{\omega_1} > a^{\omega_2}$.

In an ordered Ω-group every convex subgroup is Ω-isolated. Let G be such an ordered Ω-group that in it there is a well-ordered increasing series of convex subgroups

$$E = G_0 \subset G_1 \subset ... \subset G_\gamma \subset G_{\gamma+1} \subset ... \quad (1)$$

and the factors of this series are Ω-cyclic groups. It is not difficult to show that G will in this case be a ZA-group. Consider the group Φ of all order-preserving Ω-automorphisms of the group G. The series (1) will be Φ-admissible and, consequently, Φ is a triangular group of automorphisms. The group of automorphisms Φ can be ordered.

We give the main steps of the proof. It is not difficult to show that each convex subgroup G_γ is distinguished in $G_{\gamma+1}$ by a semidirect factor: $G_{\gamma+1} = G_\gamma \lambda B_{\gamma+1}$, where $B_{\gamma+1}$ is an Ω-cyclic group. We shall say that an automorphism $\varphi \in \Phi$ belongs to the subgroup $G_{\gamma+1}$ if $G_{\gamma+1}$ is the first of the subgroups in the series (1) on which the automorphism φ acts non-identically. If the automorphism φ belongs to the subgroup $G_{\gamma+1}$, then for all positive elements g such that $g \in G_{\gamma+1}$, but $g \notin G_\gamma$, at the same time either $\varphi(g) > g$, or $\varphi(g) < g$. In the first case we shall consider the automorphism positive ($\varphi > \varepsilon$, where ε is the identity automorphism), and in the second—negative. It is not difficult to verify that the group Φ is ordered in this way.

Denote by Σ the Φ-centralizer of the series (1). Σ is an invariant subgroup in Φ. Of two automorphisms $\sigma, \psi \in \Sigma$, the greater is the one which belongs to the subgroup with the smaller index. If, however, σ and ψ belong to the subgroup $G_{\gamma+1}$, then the greater of them will be the one which induces the identity automorphism in the factor group $G_{\gamma+1}/G_\beta$ ($\beta \leqslant \gamma$) with the greater index β. From what has been said it follows that the subgroup $\Sigma^\beta_{\gamma+1}$ of Σ, consisting of all those automorphisms which induce the identity automorphism in G_γ and $G_{\gamma+1}/G_\beta$, is convex in Σ. All such possible subgroups form a convex series in Σ. If the series (1) is finite, then on the basis of the theorem one may assert that the factors of the convex series in Σ obtained in this way are isomorphic to subgroups of the factors of the series (1).

Received
9 I 1962

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Note: Figure translations are in progress. See original paper for figures.