MATHEMATICS

1962

SovietRxiv

Abstract

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MATHEMATICS

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EMBEDDING ALGEBRAS IN ALGEBRAICALLY CLOSED ALGEBRAS

(Presented by Academician A. I. Mal' tsev on 24 III 1962)

1. For brevity, in what follows arbitrary (nonassociative) algebras will be called N-algebras, commutative (anticommutative) algebras—K-algebras (respectively, AK-algebras). By Δ we shall denote any one of the following three symbols: N, K, AK.

Let A be an arbitrary fixed Δ-algebra over a once-and-for-all chosen field P; $P(x_1, \dots, x_n)$ the free Δ-algebra over P generated by $x_1, \dots, x_n$; $R = A * P(x_1, \dots, x_n)$ the free Δ-product of the algebras A and $P(x_1, \dots, x_n)$ (see ([1],[2])). Let $\{a_\alpha\}$ be some basis of the algebra A, $S = \{a_\alpha\} \cup \{x_1, \dots, x_n\}$. We order the set S, assuming that the elements of the sets $\{a_\alpha\}$ and $\{x_i\}$ are ordered by their lower indices and, in addition, putting $x_i > a_\alpha$. From the set S we form Δ-regular words (for $\Delta = K$ or $\Delta = AK$, see ([3]); an N-regular word is any nonassociative word). We order the set of Δ-regular words by declaring words of greater length to be greater than words of smaller length; if the lengths of the words $u = u_1 u_2$ and $v = v_1 v_2$ are equal, then we put $u > v$ if either $u_1 > v_1$, or $u_1 = v_1,\ u_2 > v_2$. A word v will be called Δ-special if it is regular and contains no subwords of the form $a_\alpha a_\beta$. From the definition of the free product it follows that the Δ-special words form a basis of the algebra R. The highest Δ-special word occurring in the expression of an element $v \in R$ will be called the leading word of the element v and denoted by $\overline{v}$.

Definition 1. A system of Δ-algebraic equations over an algebra A will mean a collection of equalities of the form

$$v_i(x_1, \dots, x_n) = 0, \qquad i = 1, \dots, k, \tag{1}$$

where $v_i \in A * P(x_1, \dots, x_n)$; n, k are positive integers.

The system of Δ-algebraic equations (1) will be called Δ-consistent if $v_i \notin A,\ i = 1, \dots, k$, and no word $\overline{v_i}$ is contained as a subword in the word $\overline{v_j}$, $j \neq i$.

Definition 2. A Δ-algebra U will be called Δ-algebraically closed in the strong sense if every consistent system of Δ-algebraic equations over U has at least one solution in U.

Theorem 1. *An arbitrary Δ-algebra over a field P can be embedded in a Δ-algebra over the same field P that is algebraically closed in the strong sense.*

Theorem 1 is proved by means of the methods developed in ([4]). The theorem just formulated contains, as a special case, Neumann' s theorem that every algebra can be embedded in an algebra with division ([5]).

Remark. From the proof of Lemma 1 of ([4]) it follows that every system of Δ-algebraic equations is either equivalent to a consistent (in the sense of Definition 1) system of equations, or equivalent to such a system of equations that contains an equation of the form $a = 0$, where $a \in A,\ a \neq 0$. Hence, from Theorem 1 it follows that the definition of consistency given above coincides with the usual one (a system of Δ-algebraic equations

is called consistent in the usual sense if it has a solution in a suitable Δ-extension of the algebra A.

2. As noted in (6), Neumann posed the question: for which classes of algebras can one assert that an arbitrary algebra of this class can be embedded in a division algebra of the same class? Below (a corollary of Theorem 2) examples of such classes of algebras are given.

We shall call an element $a \in A$ weakly nilpotent (c-nilpotent) if there exists an integer m, $m > 0$, such that $a^m = 0$ (for some placement of parentheses). Put further $a^{(0)} = a$, $a^{(k)} = a^{(k-1)}a^{(k-1)}$. We shall call an element a solvably nilpotent (r-nilpotent) if there exists an integer k, $k > 0$, such that $a^{(k)} = 0$.

Theorem 2. An arbitrary Δ-algebra A over a field P can be embedded in a division Δ-algebra U over the same field P, all elements of which are r-nilpotent modulo A.

For brevity, by ν we shall denote either of the two symbols c, r.

Corollary. Every ν-nil-Δ-algebra can be embedded in a division ν-nil-Δ-algebra.

3. Let now A be an arbitrary fixed Lie algebra over a field P; $P(x_1, \dots, x_n)$ the free Lie algebra over P with generators $x_1, \dots, x_n$; $R = A * P(x_1, \dots, x_n)$ the free Lie product of the algebras A and $P(x_1, \dots, x_n)$ (7).

Definition 3. A Lie algebraic equation over an algebra A is an expression of the form $v(x_1, \dots, x_n) = 0$, where $v \in A * P(x_1, \dots, x_n)$, $v \notin A$, and n is a positive integer.

Definition 4. A Lie algebra U will be called algebraically closed if every Lie algebraic equation over U has a solution in U.

Theorem 3. An arbitrary Lie algebra over a field P can be embedded in an algebraically closed Lie algebra over the same field P.

Theorem 3 is proved by means of the methods developed in (8). The theorem formulated contains, as a special case, Cohn' s theorem that every Lie algebra can be embedded in a division Lie algebra (5).

In conclusion, the author considers it his pleasant duty to express gratitude to A. I. Shirshov, under whose supervision this work was written.

Received
16 III 1962

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