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Abstract

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On the Asymptotics of the Eigenvalues of Bilinear Forms Associated with Certain Elliptic Equations Degenerating on the Boundary

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Let in Euclidean space E_n, with points $x = (x_1, \dots, x_n)$, there be given: an n-dimensional domain ω, a p-dimensional domain s, and a manifold of points k.

We shall denote: $\rho(k, x)$ is the distance of the point x from k; i is the vector index $(i_1, i_2, \dots, i_n)$, $|i| = i_1 + i_2 + \dots + i_n$; $D^i = \partial^{|i|}/\partial x_1^{i_1} \cdots \partial x_n^{i_n}$; $|s|$ is the p-dimensional volume of s.

The set of complex functions $f \in L_2(\omega)$ having a finite integral

$$\int_\omega \rho^\alpha(k, x) \sum_{|i|=m} |D^i f|^2 \, dx, \tag{1}$$

will be denoted by $W^m_{k,\alpha}(\omega)$; α is a constant from $[0, 2m)$. The closure of the set of infinitely differentiable functions finite in ω in the norm determined by the integral (1) will be denoted by $\dot{W}^m_{k,\alpha}(\omega)$.

Let an operator be given

$$a \equiv \sum_{|i|,|j|=m} D^j(a_{ij} D^i) \qquad (2m > n) \tag{2}$$

with coefficients defined on all of E_n.

The form

$$\int_\omega \rho^\alpha(k, x) \sum_{|i|,|j|=m} a_{ij} D^i f \, D^j \bar{g} \, dx,$$

considered on functions from $W^m_{k,\alpha}(\omega)$, will be denoted by $a_{k,\alpha}(f, g, \omega)$. For $\alpha = 0$ we shall write

$$a_{k,0}(f, g, \omega) \equiv a(f, g, \omega).$$

The problem of finding the eigenfunctions of the form $a_{k,\alpha}(f, g, \omega)$, considered on some subspace $V^m(\omega)$ $(\dot{W}^m_{k,\alpha}(\omega) \subseteq V^m(\omega) \subseteq W^m_{k,\alpha}(\omega))$, will be called the problem $\{a_{k,\alpha}, V^m(\omega)\}$.

For the form $a_{k,\alpha}(f, g, \omega)$, by $A_\omega(\lambda)$, $B_\omega(\lambda)$ we shall denote the number of eigenvalues of the problem $\{a_{k,\alpha}, \dot{W}^m_{k,\alpha}(\omega)\}$, respectively $\{a_{k,\alpha}, W^m_{k,\alpha}(\omega)\}$, not exceeding λ.

In what follows we assume that the coefficients of the operator (2) are continuous on the closure $\overline{\Omega}$ of some bounded n-dimensional domain Ω with sufficiently smooth boundary S, and that the matrix $\|a_{ij}\|$ is uniformly positive definite on $\overline{\Omega}$, i.e.

$$\sum_{|i|,|j|=m} a_{ij}(x)\eta_i\overline{\eta_j} \geq d \sum_{|i|=m} \frac{m!}{i_1! \cdots i_n!}|\eta_i|^2 \qquad (d > 0)$$

for all complex η_i and $x \in \overline{\Omega}$.

In the present paper the question of the asymptotics of the eigenvalues of the problem $\{a_{S,\alpha}, V^m(\Omega)\}$ is considered. In the case $\alpha = 0$, the formulas obtained are contained in the results of paper (^{5}).

It follows from the papers (^{2,3}) that for any $V^m(\Omega)$ there exists a sequence $\{u_i\} \subset V^m(\Omega)$ of functions and a corresponding sequence of numbers $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots \to \infty$ such that $a_{k,\alpha}(u_i, f, \Omega) = \lambda_i(u_i, f)$ for all $f \in V^m(\Omega)$, where (u_i, f) is the ordinary scalar product, and the u_i form a complete system in $V^m(\Omega)$.

We first carry out an estimate of the growth of the eigenvalues for domains of a special form.

Let σ be an $(n-1)$-dimensional plane domain bounded by a finite number of sufficiently smooth $(n-2)$-dimensional manifolds and having no reentrant angles. Moreover, the boundary of σ has no points at which more than $n-1$ of the manifolds bounding it intersect.

Denote

$$b \equiv \sum_{|i|,\,|j|=m} D^i(b_{ij}D^j),$$

where b_{ij} are constants and the matrix $\|b_{ij}\|$ is positive definite; $2m > n$;

$$\omega(b) = (2\pi)^{-n} \int_{b(\xi)\leqslant 1} d\xi,$$

where

$$b(\xi) = \sum_{|i|,\,|j|=m} b_{ij}\xi^{i+j},$$

ξ is a point of $E_n(\xi_1, \xi_2, \dots, \xi_n)$, $\xi^i = \xi_1^{i_1} \xi_2^{i_2} \cdots \xi_n^{i_n}$. Below, $\varepsilon(\lambda)$, with different subscripts, denotes a function tending to 0 as $\lambda \to \infty$.

By the methods of (1), using asymptotic estimates of eigenvalues for nondegenerate elliptic operators (5), one obtains the following assertions.

Lemma 1. Let Π be a cylinder with points $x = (x', x_n)$; $x' = (x_1, \dots, x_{n-1}) \in \sigma$, $0 \leqslant x_n \leqslant l$. For the form $b(f, g, \Pi)$ the relations are valid:

$$A_\Pi(\lambda) \geqslant |\Pi|\, \omega(b) \lambda^{n/2m} (1 + \varepsilon_1(l^{-1}))(1 + \varepsilon_{1,0}(\lambda l^{2m})),$$

$$B_\Pi(\lambda) \leqslant |\Pi|\, \omega(b) \lambda^{n/2m} (1 + \varepsilon_2(l^{-1}))(1 + \varepsilon_{2,0}(\lambda l^{2m})).$$

Lemma 2. Let Π be the cylinder $x' \in \sigma$, $0 \leqslant x_n \leqslant l = (\gamma/\lambda)^{\frac{1}{2m-\alpha}}$. For the problem $\{b_{\sigma,\alpha}, W^m_{\sigma,\alpha}(\Pi)\}$, for every $\gamma > 0$ the estimate holds

$$B_\Pi(\lambda) \leqslant |\sigma| K(\gamma) \lambda^{\frac{n}{2m} + \frac{\frac{\alpha}{2m} - 1}{2m-\alpha}} (1 + \varepsilon_3(l^{-1})),$$

where

$$K(\gamma) = K_0 \gamma^{-\frac{n-1}{2m-2}},$$

and K_0 is a constant.

These assertions make it possible, by the methods of R. Courant (1), with the use of the techniques of (4), to obtain successively the following theorems.

Theorem 1. Let Π be the cylinder $x' \in \sigma$, $0 \leqslant x_n \leqslant l$. For the form $b_{\sigma,\alpha}(f, g, \Pi)$ the relations are valid:

$$A_\Pi(\lambda),\ B_\Pi(\lambda) = M(\lambda) \omega(b) |\sigma| \lambda^{\frac{n}{2m} + \frac{\beta-1}{2m-\alpha}} (1 + \varepsilon_4(l^{-1})) + \cdots \quad \text{for } \beta > 1,$$

$$A_\Pi(\lambda),\ B_\Pi(\lambda) = \frac{\omega(b)}{2m-\alpha} \sigma |\lambda|^{\frac{n}{2m}} \ln \lambda (1 + \varepsilon_5(l^{-1})) + \cdots \quad \text{for } \beta = 1,$$

$$A_\Pi(\lambda),\ B_\Pi(\lambda) = \omega(b) |\sigma| \lambda^{\frac{n}{2m}} (1 + \varepsilon_6(l^{-1})) \int_0^l \frac{dx}{x^\beta} + \cdots \quad \text{for } \delta < 1.$$

Here and below, $\beta = \alpha n / 2m$; the dots replace terms of lower order with respect to λ; $M(\lambda)$ is an unknown function of λ, bounded below and above by positive constants.

Theorem 2. For the eigenvalues of the form $a_{S,\alpha}(f, g, \Omega)$ the following asymptotic formulas hold:

$$A_\Omega(\lambda),\ B_\Omega(\lambda) = M(\lambda) \lambda^{\frac{n}{2m} + \frac{\beta-1}{2m-\alpha}} \int_S w(a)\, dS + \dots \qquad \text{for } \beta > 1,$$

$$A_\Omega(\lambda),\ B_\Omega(\lambda) = \frac{1}{2m-\alpha} \lambda^{\frac{n}{2m}} \ln \lambda \int_S w(a)\, dS + \dots \qquad \text{for } \beta = 1,$$

$$A_\Omega(\lambda),\ B_\Omega(\lambda) = \lambda^{\frac{n}{2m}} \int_\Omega \frac{w(a)}{\rho^\beta(S,x)}\,dx + \ldots \qquad \text{for } \beta < 1.$$

Remark 1. The formulas of Theorem 2 remain valid without change also for the more general form, taking real values,

$$a_{S,\alpha}(f,g,\Omega) + \int_\Omega \sum_{|i|,\,|j|\le m;\ |i|+|j|<2m} r_{ij}\rho^{\alpha_{ij}}(S,x) D^i f\, \overline{D^j g}\, dx,$$

where r_{ij} are bounded functions measurable in Ω; α_{ij} are constants satisfying the condition $\alpha_{ij} > \alpha - (2m - |i| - |j|)$.

Remark 2. The asymptotic estimates obtained are valid for the eigenvalues of the problems $\{a_{S,\alpha}, V^m(\Omega)\}$.

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References

1. R. Courant, D. Hilbert, *Methods of Mathematical Physics*, 1, Ch. 6; 2, Ch. 7, Moscow-Leningrad, 1951.
2. M. I. Vishik, *Matem. sbornik*, 35 (77), 3, 513 (1954).
3. S. L. Sobolev, *Some Applications of Functional Analysis in Mathematical Physics*, Leningrad, 1950.
4. I. A. Solomeshch, *Matem. sbornik*, 54 (96), 3, 295 (1961).
5. G. Ehrling, *Math. Scand.*, 2, fasc. 2, 267 (1954).

Note: Figure translations are in progress. See original paper for figures.