MATHEMATICS

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1962

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Abstract

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MATHEMATICS

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ON BENDING AND THE UNIQUE DETERMINACY OF SURFACES OF POSITIVE CURVATURE WITH BOUNDARY

(Presented by Academician I. G. Petrovskii, 22 XII 1961)

The problem of the unique determinacy of surfaces of positive curvature with boundary is considered under certain boundary conditions. The boundary conditions are of a rather general nature and are formulated precisely below. In particular, it is proved that two isometric surfaces of positive curvature whose boundaries have equal curvatures are congruent or symmetric. We also investigate the question of bending surfaces with preservation of the principal curvatures (principal directions) along the boundary. It is shown, for example, that all surfaces whose boundaries contain no umbilic points can be divided into two classes. The first consists of surfaces that admit no isometric transformations preserving the principal curvatures (principal directions) along the boundary; the second consists of surfaces that admit even continuous bendings under this condition. It is shown that the division of surfaces into classes depends essentially on the net of curvature lines along the boundary of the surface under consideration. All the theorems proved remain valid for infinitesimal bendings. Related questions are treated in the work of K. M. Belov (^{1}) and in our note (^{2}).

1. We consider simply connected surfaces of strictly positive Gaussian curvature up to the boundary. We assume that the surfaces belong to the class $D_{3,p}$, $p > 2$ (the radius vector of the surface $\mathbf{r}(u, v)$ has three generalized derivatives in the sense of Sobolev, summable with power p). The boundary of the surface is assumed to be a simple smooth closed curve of class C^1_μ, $0 < \mu \leq 1$. If a surface S is transformed isometrically into a surface S^*, then we require that $S^* \in D_{3,p}$, $p > 2$.

2. To formulate the results obtained, we introduce some notation. Let a surface S with boundary L be given, and let the boundary and the surface be oriented so that, when traversing L, the surface lies on the left. Suppose further that on S there is given a continuous field of directions R having no singular points on L. At some point P of the curve L, assign a direction of the field by an arrow (choosing arbitrarily one of the two possibilities). Denote by ϑ the angle formed by this arrow and the directed tangent to the curve L at the point P. We measure the angle from the arrow

to the tangent counterclockwise. Choose $\vartheta < 2\pi$. The value of the angle thus obtained is then continued continuously, causing the point P to move along the curve L in the direction that leaves S on the left. The change of the angle ϑ in one traversal of the curve L will be denoted by $v_R(S) \cdot \pi$. The number $v_R(S)$ will be called the index of the surface S with respect to the field of directions R. It can be proved (^{3}) that $v_R(S)$ is a topological invariant. Obviously, if the field of directions R_0 has no singular points, then $v_{R_0}(S) = +2$.

Let x be an isolated singular point of the field of directions R. Surround it by a smooth curve γ containing within it the single singular point x and passing through no other singular points of the field of directions R. The index of the singular point x is defined to be the number $v_R(x) = v_R(\tilde{\gamma})$, where $\tilde{\gamma}$ is the part of the surface S,

bounded by a curve γ. If the field of directions R_n has n isolated singular points x_i, then it is easy to show $^{(4)}$ that

$$v_{R_n}(S) = \sum_{i=1}^{n} v_{R_n}(x_i) - 2(n-1). \tag{*}$$

3. Let S be a surface of positive curvature with boundary L, and let R be some continuous field of directions. Denote by k_{nR} and τ_{gR}, respectively, the normal curvature of the surface and the geodesic torsion in the direction R.

Theorem 1. If $v_R(S) < 2$, then every isometric transformation of the surface S that does not change along the boundary L the normal curvature k_{nR} (the geodesic torsion τ_{gR}) is a motion or a mirror reflection. If $v_R(S) \geqslant 2$, then there even exists a continuous bending of the surface with preservation of k_{nR} (τ_{gR}) along the boundary L.

In the case $v_R(S) \geqslant 2$ a stronger assertion holds.

Theorem 2. Let a function $\sigma(s) \in C_\alpha(L)$, $0 < \alpha < 1$, be given on the contour L, where s is the arc length of L, and suppose $\|\sigma\|_\alpha < \varepsilon$ (ε is a sufficiently small positive number depending on the surface). Suppose, further, that $v_R(S) \geqslant 2$. Then there exists a continuous bending of the surface S such that the increment of the normal curvature k_{nR} (of the geodesic torsion τ_{gR}) under the bending along the boundary is equal to $\sigma(s)$.

We single out a special case of Theorem 1 (the field of directions R along the boundary L coincides with L) in the form of the following theorem.

Theorem 3. A simply connected piece of a surface of positive curvature admits no isometric transformations preserving the curvature of the boundary.

The requirement of smoothness (and not piecewise smoothness) of the boundary in Theorems 1-3 is essential. For example, there exists a piece of a spherical surface with a piecewise-smooth boundary (the angles at the corner points are nonzero) that admits a nontrivial isometric transformation under which the geodesic curvature of the boundary does not change at any point. Indeed, take a hemisphere S and bend it so that on the bent surface S_t the net of curvature lines forms a regular net. According to (5), such a bending is possible. From the surface S_t cut out a curvilinear quadrilateral f_t whose sides are lines of curvature. It is obvious that all angles at the corner points of the boundary are nonzero and that along the boundary the geodesic curvature is zero. Now cut out from the hemisphere S a quadrilateral f isometric to f_t. The piece f is not congruent to f_t, although the geodesic curvature of the boundary of the surface f is also equal to zero.

Similarly one can show that Theorem 3, generally speaking, is not true for surfaces with piecewise-smooth boundary.

The condition of Theorem 3 for surfaces of positive curvature can be realized, for example, by gluing the surface S to some regular surface F so that these surfaces nowhere touch along the line of gluing and the angle of gluing under an isometric transformation of the surface $S + F$ remains unchanged.

We shall say that a surface S is bordered by an asymptotic strip if its boundary curve is glued to an asymptotic strip whose base is this curve. It is known (6) that in this case a surface S of positive curvature is rigid. The following also holds.

Theorem 4. A piece of a surface of positive curvature bordered by an asymptotic strip is uniquely determined.

Remark. Theorem 3 is also valid for a general convex surface whose boundary consists of a finite number of plane curves of class C^2. Theorem 4 is valid for caps of nonnegative curvature with plane boundary.

Theorem 5. If $v_R(S) < 4$, then every isometric transformation of the surface S that does not change along the boundary the product $k_{nR} \cdot k_{nL}$ (where k_{nL} – normal curvature of the surface in the direction of the edge), is a motion or a mirror reflection. If $v_R(S) \geqslant 4$, then there even exist discontinuous bendings of the surface S preserving along the edge $k_{nR} \cdot k_{nL}$.

4. Let now the field of directions R on S be the field of principal directions (we shall denote it by). The following theorem is valid.

Theorem 6. If $v\ (S) < 2$ and the edge L contains no umbilic points, then every isometric transformation of the surface S preserving the principal curvatures (principal directions) along L is a motion or a mirror reflection. If $v\ (S) \geqslant 2$, then there exists a continuous bending of S preserving the principal curvatures (principal directions) along the edge.

In particular, if the surface S contains n umbilic points x_i, not lying on the edge, then, according to formula $(*)$,

$$v\ (S) = \sum_{i=1}^{n} v\ (x_i) - 2n + 2.$$

Consequently, $v\ (S) < 2$, if $\sum_{i=1}^{n} v\ (x_i) < 2n$; $v\ (S) \geqslant 2$, if

$$\sum_{i=1}^{n} v\ (x_i) \geqslant 2n.$$

The index $v\ (x_i)$ of the umbilic point x_i may be computed directly ([3]). For example, the index of an umbilic point of a surface of revolution is equal to 0, that of a triaxial ellipsoid is equal to +1; in general, the index of an umbilic point is $\geqslant -2$.

Let us indicate one more result following from Theorem 6. Suppose that on an ovaloid S there is given some simple smooth closed curve L, not passing through umbilic points of the surface. Cut S along L into two simply connected pieces S_1 and S_2, $S = S_1 + S_2$. Then $v\ (S_1) = -v\ (S_2)$, i.e. if S_1 admits continuous bendings with preservation of the principal curvatures (principal directions) along the edge, then S_2 does not admit even isometric transformations under this condition on the edge.

Suppose that there are umbilic points on the edge L of the surface S. Assign directions at these points so that the field of principal directions along L is continuous and has no singular points. Denote the obtained field of directions by $\tilde{}$. From Theorem 1 it follows:

Theorem 7. If $v_{\sim}(S) < 2$, then every isometric transformation of the surface S preserving along the edge the principal directions at a non-umbilic point and the indicated direction of an umbilic point is a motion or a mirror reflection. If $v_{\sim}(S) \geqslant 2$, then the surface S admits continuous bendings under the conditions indicated above.

The proof of the formulated Theorems 1-7 is based on the results of papers ([2],[5]).

In conclusion I express my gratitude to G. S. Barkhin for guidance of this work.

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Received
19 XII 1961

CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.