# Mathematics

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1962

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**Abstract**

**Full Text**

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# On a Conjecture of V. A. Steklov

*(Presented by Academician S. N. Bernstein on 18 IX 1961)*

1. V. A. Steklov put forward the conjecture ([1]):

If polynomials are orthonormal on an interval of the real axis with respect to some weight, then positivity of the weight is necessary and sufficient for the boundedness of the entire orthonormal system.

The present note is a certain attempt to approach a proof of this conjecture; for completeness we give all the results obtained by us in this direction in recent times.

2. Let the polynomials $\{\varphi_n(z)\}_0^\infty$ be orthonormal on the circle $z = e^{i\theta}$, $0 \leq \theta \leq 2\pi$, with respect to the distribution $d\sigma(\theta)$.

**Theorem 1.** *If $\sigma'(\theta_0)$ exists, then, for the inequality $\sigma'(\theta_0) > 0$ to hold, it is sufficient that the following three conditions be satisfied simultaneously:*

$$\lim_{n\to\infty} |\varphi_n(e^{i\theta_0})| < \infty; \tag{1}$$

$$\int_0^h |dt\{\sigma(\theta_0 + t) - \sigma(\theta_0 - t) - 2t\sigma'(\theta_0)\}| = o(h); \tag{2}$$

$$\overline{\lim}_{n\to\infty} |a_n| < 1, \tag{3}$$

*where*

$$\sqrt{1 - |a_n|^2}\varphi_{n+1}(z) = z\varphi_n(z) - \overline{a_n}\varphi_n^*(z), \qquad \varphi_n^*(z) = z^n \overline{\varphi_n\left(\frac{1}{\overline{z}}\right)},$$

$$|a_n| < 1 \qquad (n = 0, 1, 2, ...).$$

Let us note first of all that the theorem remains valid if one requires the existence only of the generalized symmetric derivative $\sigma^{(1)}(\theta_0)$. Further, condition (1) is equivalent to the existence of a subsequence $\{\varphi_{n_i}(e^{i\theta})\}$ for which

$$|\varphi_{n_i}(e^{i\theta_0})| \leq M \qquad (i = 1, 2, \ldots); \tag{1'}$$

in this case condition (2) may be replaced by the less restrictive condition

$$\int_0^h |dt\{\sigma(\theta_0 + t) - \sigma(\theta_0 - t) - 2t\sigma'(\theta_0)\}| \leq C_1 h, \qquad C_1\sqrt{M} < \left(\frac{2}{\pi} + \frac{\pi}{2}\right)^{-1/2}. \tag{4}$$

If the function $\sigma(\theta)$ is absolutely continuous on the interval $[\theta_0 - \varepsilon, \theta_0 + \varepsilon]$, $\varepsilon > 0$, then condition (2) is replaced by the condition

$$\int_0^h |p(\theta_0 + t) + p(\theta_0 - t) - 2p(\theta_0)|\, dt = o(h), \qquad p(\theta) = \sigma'(\theta), \tag{2'}$$

which is always satisfied if $\theta_0$ is a Lebesgue point of the function

$p(\theta)$; in exactly the same way condition (4) is replaced by the condition

$$\int_0^h |p(\theta_0 + t) + p(\theta_0 - t) - 2p(\theta_0)|\, dt \leq C_1 h, \qquad C_1\sqrt{M} < \left(\frac{2}{\pi} + \frac{\pi}{2}\right)^{-1/2}. \tag{4'}$$

Finally, condition (3) can be replaced by the more restrictive condition

$$\int_0^{2\pi} \lg \sigma'(\theta)\, d\theta > -\infty, \tag{3'}$$

which, however, has the advantage over (3) that it is imposed on the original function $\sigma(\theta)$, and not on the parameters $\{a_n\}$.

3. **Theorem 2.** *If the inequalities*

$$|\varphi_{n_i}(e^{i\theta})| \leq M, \qquad \alpha \leq \theta \leq \beta, \quad 0 \leq \alpha, \beta \leq 2\pi \quad (i = 1, 2, \ldots), \tag{5}$$

*hold, then for any two points of discontinuity* $\theta_1, \theta_2$ *of the function* $\sigma(\theta)$ *the inequalities*

$$\sigma(\theta_2) - \sigma(\theta_1) \geq m(\theta_2 - \theta_1), \qquad m = \frac{1}{M^2}, \quad \alpha \leq \theta_1 < \theta_2 \leq \beta \tag{6}$$

*hold.*

In proving the converse theorem, one has to distinguish two cases: $[\alpha, \beta] \subset [0, 2\pi]$ and $[\alpha, \beta] = [0, 2\pi]$.

**Theorem 3.** *If* $[\alpha, \beta] \subset [0, 2\pi]$*, then condition (3) together with (6) is sufficient for the inequalities*

$$\overline{\lim_{n\to\infty}} |\varphi_n(e^{i\theta})| < \infty, \qquad \alpha + \varepsilon \le \theta \le \beta - \varepsilon, \quad \varepsilon > 0; \tag{7}$$

$$|\varphi_n(e^{i\theta})| \le M, \qquad \theta \in [\alpha', \beta'] \subset [\alpha + \varepsilon, \beta - \varepsilon] \quad (n = 0, 1, ...). \tag{8}$$

**Theorem 4.** *If (6) is true for all* $0 \le \theta_1 < \theta_2 \le 2\pi$*, then inequality (7) is true for all points of the interval* $[0, 2\pi]$*, and inequality (8) is true on some interior interval* $[\alpha', \beta'] \subset [0, 2\pi]$*.*

In the case under consideration, (6) implies (3′), whence, as was said, (3) follows –therefore in the formulation of Theorem 4 condition (3) may be omitted.

**Remark.** Theorems 2 and 4 prove the validity of V. A. Steklov' s conjecture for the whole interval $[0, 2\pi]$; however, we assert boundedness of the orthonormal system at each point of the interval $[0, 2\pi]$, while uniform boundedness only on some interior interval $[\alpha', \beta'] \subset [0, 2\pi]$. Theorems 2 and 3 prove the validity of V. A. Steklov' s conjecture for the case of an interval $[\alpha, \beta] \subset [0, 2\pi]$; however, in Theorem 3 we had to introduce the additional restriction (3), without which one could assert only the boundedness of the arithmetic means

$$\left\{ \frac{1}{n+1} \sum_{k=0}^{n} |\varphi_k(e^{i\theta})|^2 \right\}.$$

Finally, condition (3) in Theorem 1 may be omitted if in (1) one replaces lim by $\overline{\lim}$*.

4. G. Szegő' s formula

$$p_n(x) = \frac{\varphi_{2n}(z) + \varphi^*_{2n}(z)}{2\pi\sqrt{1 - a_{2n-1}}}\, z^{-n}, \qquad x = \frac{1}{2}\left(z + \frac{1}{z}\right), \tag{9}$$

allows one to express the polynomials $\{p_n(x)\}$, orthonormal on the interval

* It is possible that condition (6) implies the validity of (3), but we have not yet been able to prove this.

$[-1, +1]$ with respect to the weight $d\psi(x)$, through the polynomials $\{\varphi_n(z)\}$ considered above, under the condition

$$\sigma(\theta) = \begin{cases} -\psi(\cos\theta), & 0 \leqslant \theta \leqslant \pi, \\ \psi(\cos\theta), & \pi \leqslant \theta \leqslant 2\pi. \end{cases} \tag{10}$$

For $-1 \leqslant x \leqslant 1$ we have

$$|p_n(x)| \leqslant \frac{|\varphi_{2n}(e^{i\theta})|}{\pi\sqrt{1-a_{2n-1}}} \quad (n = 1, 2, ...); \tag{11}$$

thus, under condition (3), the boundedness of $|\varphi_{2n}(e^{i\theta})|$ entails the boundedness of $|p_n(x)|$.

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Received
9 IX 1961

## CITED LITERATURE

1. V. A. Steklov, *Izv. Rossiisk. Akad. nauk*, **15**, 281 (1921).

*Note: Figure translations are in progress. See original paper for figures.*