ON EXPANSION IN EIGENFUNCTIONS OF A NON-SELF-ADJOINT DIFFERENTIAL OPERATOR OF EVEN ORDER IN THE SPACE OF VECTOR FUNCTIONS

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Abstract

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M. G. GIMADISLAMOV

ON EXPANSION IN EIGENFUNCTIONS OF A NON-SELF-ADJOINT DIFFERENTIAL OPERATOR OF EVEN ORDER IN THE SPACE OF VECTOR FUNCTIONS

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In the present paper we study expansion in eigenfunctions of a non-self-adjoint system of differential equations of arbitrary even order on the half-axis $[0, \infty)$.

Consider a system of differential expressions of order $2n$, which we write in the form:

$$l(y) = y^{(2n)} + P_2(x)y^{(2n-2)} + P_3(x)y^{(2n-3)} + \dots + P_{2n}y, \tag{1}$$

where $y(x) = (y_1(x), \dots, y_k(x))$ is a vector function; $P_\nu(x)$, $\nu = 2, \dots, 2n$, are complex-valued matrix functions of order k, summable on the interval $[0, \infty)$.

Denote by D the set of all vector functions $y(x) \in L_k^2(0, \infty)$ such that: 1) the derivatives $y^{(\nu)}(x)$, $\nu = 1, 2, \dots, 2n-1$, exist and are absolutely continuous on every finite interval $[0, b]$, $b > 0$; 2) $l(y) \in L_k^2(0, \infty)$.

Denote by D_A the set of all vector functions $y(x) \in D$ satisfying the boundary conditions

$$u_\nu(y) = A_{\nu,2n-1}y^{(2n-1)}(0) + A_{\nu,2n-2}y^{(2n-2)}(0) + \dots + A_{\nu,0}y(0) = 0,$$

$$\nu = 1, 2, \dots, n, \tag{2}$$

where $A_{\nu,j}$ are complex matrices of order k. We define the operator L_A as follows: its domain of definition is D_A, and for $y \in D_A$

$$L_{Ay} = l(y). \tag{3}$$

The operator L_A^*, adjoint to L_A, is constructed in an analogous way for the differential expression adjoint to (1),

$$l^*(z) = z^{(2n)} + (P_2^*(x)z)^{(2n-2)} - (P_3^*(x)z)^{(2n-3)} + \dots + P_{2n}^* z \tag{4}$$

and for the boundary conditions adjoint to (2),

$$v_\nu(z) = B_{\nu,2n-1} z^{(2n-1)}(0) + B_{\nu,2n-2} z^{(2n-2)}(0) + \dots + B_{\nu,0} z(0) = 0,$$

$$\nu = 1, \dots, n. \tag{5}$$

Put $\rho^{2n} = -\lambda$. Let $\omega_1, \dots, \omega_{2n}$ be the roots of degree $2n$ of -1; divide the complex ρ-plane into $2n$ equal sectors S_k, $k = 0, 1, \dots, 2n-1$, defined by the inequality

$$\frac{k\pi}{n} < \arg\rho < \frac{(k+1)\pi}{n}.$$

In each sector S_k one can choose an ordering of the numbers $\omega_1, \dots, \omega_{2n}$ such that, for $\rho \in S_k$,

$$\mathrm{Re}(\rho\omega_1) \leqslant \mathrm{Re}(\rho\omega_2) \leqslant \dots \leqslant \mathrm{Re}(\rho\omega_{2n}).$$

Denote by T_k and T_{k-1} the boundaries of the sector S_k. Let the matrix functions $P_\nu(x)$ satisfy the additional condition

$$e^{\varepsilon_2 x}\,|P_\nu(x)| \leqslant C_\nu. \tag{6}$$

Consider the matrix equation

$$Y^{(2n)} + P_2(x)Y^{(2n-2)} + \dots + P_{2n}(x)Y = \lambda Y. \tag{7}$$

It can be shown that equation (7) has linearly independent solutions $Y_j(x,\rho)$, $j = 1, 2, \dots, 2n$, holomorphic with respect to ρ for $\rho \in S_j$ and having the asymptotics:

$$\text{as } x \to \infty \qquad Y_j^{(\nu)}(x,\rho) = \rho^\nu e^{\rho\omega_j x}\left[\omega_j^\nu \cdot 1 + o(1)\right] \tag{8}$$

uniformly with respect to $\rho \in S_j$,

$$\text{as } \rho \to \infty \qquad Y_j^{(\nu)}(x,\rho) = \rho^\nu e^{\rho\omega_j x}\left[\omega_j^\nu \cdot 1 + O\left(\frac{1}{\rho}\right)\right] \tag{9}$$

uniformly with respect to $x \in [0, \infty)$.

Similarly, we construct solutions $Z_j(x,\rho)$, $j = 1, \dots, 2n$, for the matrix equation

$$Z^{(2n)} + (ZP_2(x))^{(2n-2)} - (ZP_3(x))^{(2n-3)} + \dots + ZP_{2n}(x) = \lambda Z, \qquad (10)$$

normalized in a definite manner.

Denote

$$A(\rho) = \begin{vmatrix} u_1(Y_1) & \cdots u_1(Y_{n-1}) & u_1(Y_n) \\ \cdots & \cdots & \cdots \\ u_n(Y_1) & \cdots u_n(Y_{n-1}) & u_n(Y_n) \end{vmatrix},$$

$$\widetilde{A}(\rho) = \begin{vmatrix} u_1(Y_1) & \cdots u_1(Y_{n-1}) & u_1(Y_{n+1}) \\ \cdots & \cdots & \cdots \\ u_n(Y_1) & \cdots u_n(Y_{n-1}) & u_n(Y_{n+1}) \end{vmatrix}. \qquad (11)$$

We shall assume, for simplicity, that $A(\rho) \neq 0$, $\widetilde{A}(\rho) \neq 0$ for $\rho \in T_k$, and that the eigenvalues of the operator L_A are simple.

Theorem 1. The spectrum of the operator L_A is continuous, for even n, on the positive semiaxis (for odd n, respectively, on the negative semiaxis) and is discrete in the entire complex λ-plane. The eigenvalues form a finite set. For values of λ not belonging to the spectrum, the resolvent $(L_A - \lambda 1)^{-1}$ of the operator L_A is a bounded integral operator with kernel $K(x, t, \lambda)$ satisfying the conditions

$$\int_0^\infty |K(x,t,\lambda)|^2 \, dt < \infty, \qquad \int_0^\infty |K(x,t,\lambda)|^2 \, dx < \infty.$$

Consider the auxiliary boundary-value problem on the interval $[0, b]$:

$$l(y) = \lambda y; \qquad u_\nu(y) = 0, \quad \nu = 1, 2, \dots, n;$$

$$u_{\mu b}(y) = y^{(\mu-1)}(b) = 0, \quad \mu = 1, 2, \dots, n. \qquad (12)$$

For sufficiently large b, to each eigenvalue $\lambda_1, \dots, \lambda_r$ of the operator L_A there corresponds exactly one eigenvalue $\lambda_1(b), \dots, \lambda_r(b)$ of the boundary-value problem (12), such that $\lambda_k(b) \to \lambda_k$ as $b \to \infty$. All

the remaining eigenvalues of the auxiliary boundary-value problem as $b \to \infty$ satisfy the following asymptotic relations:

$$\lambda = -\left(\rho_m^{(j)}\right)^{2n}, \qquad \rho_m^{(j)} \omega_n = \frac{m\pi i}{b} + \frac{1}{2b} \ln \xi_j \left(\frac{m\pi i}{\omega_n b}\right) + \frac{1}{b} O(1),$$

$$j = 1, 2, \dots, k, \tag{13}$$

uniformly with respect to ρ in the domain $|\mathrm{Re}(\rho\omega_n)| \leqslant \varepsilon_1,\ \varepsilon_1 < \varepsilon_2,\ 0 \leqslant |\rho| \leqslant N$, $\xi_j(\rho)$ are the roots of the algebraic equation of order k with respect to ξ,

$$\theta_k \xi^k + \theta_{k-1}\xi^{k-1} + \dots + \theta_0 = 0, \tag{14}$$

where

$$\theta_k = \begin{vmatrix} u_1(Y_1) & \cdots & u_1(Y_{n-1}) & u_1(Y_n) \\ \cdot & \cdots & \cdot & \cdot \\ u_n(Y_1) & \cdots & u_n(Y_{n-1}) & u_n(Y_n) \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \cdot & \cdots & \cdot & \cdot \\ \omega_n^{n-1}\cdot 1 & \omega_{n+2}^{n-1}\cdot 1 & \cdots & \omega_{2n}^{n-1}\cdot 1 \end{vmatrix},$$

$$\theta_0 = \begin{vmatrix} u_1(Y_1) & \cdots & u_1(Y_{n-1}) & u_1(Y_{n+1}) \\ \cdot & \cdots & \cdot & \cdot \\ u_n(Y_1) & \cdots & u_n(Y_{n-1}) & u_n(Y_{n+1}) \end{vmatrix} \cdot \begin{vmatrix} 1 & \cdots & 1 \\ \cdot & \cdots & \cdot \\ \omega_{n+1}^{n-1}\cdot 1 & \cdots & \omega_{2n}^{n-1}\cdot 1 \end{vmatrix}.$$

Let $y_j(x)$ be an eigenfunction of the operator L_A corresponding to the eigenvalue $\lambda_j,\ j = 1, 2, \dots, r$; then

$$y_j(x) = \left[-\sum Y_i(x,\rho)\, T_{i\nu} u_\nu(Y_n) + Y_n(x)\right] c_j, \tag{15}$$

where c_j is a k-dimensional vector and

$$\begin{pmatrix} T_{11} & \cdots & T_{1,n-1} \\ \cdot & \cdots & \cdot \\ T_{n-1,1} & \cdots & T_{n-1,n-1} \end{pmatrix} \begin{pmatrix} u_1(Y_1) & \cdots & u_1(Y_{n-1}) \\ \cdot & \cdots & \cdot \\ u_{n-1}(Y_1) & \cdots & u_{n-1}(Y_{n-1}) \end{pmatrix} = 1.$$

Let $z_j(t),\ j = 1, 2, \dots, r$, be the eigenfunctions of the adjoint boundary-value problem; then as $b \to \infty$

$$\frac{y_j(x,b) z_j^*(t,b)}{\displaystyle\int_0^b (y_j, z_j)\, dx} = \frac{y_j(x) z_j(t)}{\displaystyle\int_0^\infty (y_j, z_j)\, dx} + o(1) \tag{16}$$

uniformly with respect to x, t in the square $0 \leqslant x, t \leqslant c,\ c > 0$. The eigenfunctions of the boundary-value problem (12), corresponding to the eigenvalues (13), have the following asymptotics as $b \to \infty$:

$$y\left(x,\rho_m^{(j)}\right)=\left\{-\sum_{i,k=1}^{n-1}Y_i\left(x,\rho_m^{(j)}\right)T_{ik}\left[u_k(Y_n)-a\xi_j\left(\rho_m^{(j)}\right)u_k(Y_{n+1})\right]+\right.$$

$$\left.+Y_n\left(x,\rho_m^{(j)}\right)-a\xi_j\left(\rho_m^{(j)}\right)Y_{n+1}\rho\left(x,\rho_m^{(j)}\right)\right\}r_j\left(\rho_m^{(j)}\right)+o(1), \tag{17}$$

where $a\neq 0$ is a constant depending only on $\omega_n,\omega_{n+1},\dots,\omega_{2n}$; $r_j(\rho)$ is a uniquely determined k-dimensional vector.

The eigenfunctions of the boundary-value problem adjoint to problem (12) are constructed with the aid of solutions of equation (10) and have the form

$$z\left(x,\rho_m^{(j)}\right)=\left\{-\sum_{i,k=1}^{n-1}z_i^*\left(x,\rho_m^{(j)}\right)T'_{ik}\left[v_k(z_n^*)-\overline{a}\,\overline{\xi}'_j\,v_k(z_{n+1}^*)\right]+\right.$$

$$\left.+z_n^*\left(x,\rho_m^{(j)}\right)-\overline{a}\,\overline{\xi}'_j\left(\rho_m^{(j)}\right)z_{n+1}^*\left(x,\rho_m^{(j)}\right)\right\}r'_j\left(\rho_m^{(j)}\right)+o(1), \tag{18}$$

where $T'_{ik},\ \xi'_j,\ r'_j$ are constructed in the same way as $T_{ik},\ \xi_j,\ r_j$.

Further,

$$\frac{1}{b}\int_0^b(y(x,\rho_m^{(j)}),\,z(x,\rho_m^{(j)}))\,dx=-a[\xi_j+\xi'_j](r_j,r'_j)+o(1) \tag{19}$$

as $b\to\infty$.

Let $K_b(x,t,\lambda)$ be the kernel of the resolvent of the auxiliary boundary-value problem (12); then, as $b\to\infty$,

$$K_b(x,t,\lambda)=K(x,t,\lambda)+o(1) \tag{20}$$

uniformly with respect to x,t in every finite square $0\le x,t\le c,\ c>0$.

Theorem 2. *Suppose that conditions (6) and (11) are satisfied, and suppose that the operator L_A has only simple eigenvalues $\lambda_1=-\rho_1^{2n},\dots,\lambda_r=-\rho_r^{2n}$; let $y_1,\dots,y_r$ be the corresponding eigenfunctions. Let $K(x,t,\lambda)$ be the kernel of the resolvent of the operator L_A. Then, for any point λ not belonging to the spectrum of the operator L_A,*

$$K(x,t,\lambda)=\sum_{j=1}^{r}\frac{y_j(x)z_j^*(t)}{\displaystyle\int_0^\infty(y_j,z_j)\,dx}+\frac{\omega_n}{\pi i}\int_{T_k}\sum_{j=1}^{k}\frac{\tilde{y}_j(x,\rho)\,\tilde{z}_j^*(t,\rho)}{(\rho^{2n}+\lambda)a(\xi_j+\xi'_j)(r_j,r'_j)}\,d\rho, \tag{21}$$

where

$$\tilde{y}_j(x,\rho) = \left\{ -\sum_{i,k=1}^{n-1} Y_i(x,\rho)\, T_{ik}[u_k(Y_n) - a\xi_j u_k(Y_{n+1})] + Y_n(x,\rho) - a\xi_{jY_{n+1}}(x,\rho) \right\} r_j,$$

$$\tilde{z}_j^*(t,\rho) = r_j'^* \left\{ -\sum_{l,\mu=1}^{n-1} [v_\mu(z_n) - a\xi_j' v_\mu(z_{n+1})] T'_{\mu,l} z_l(t,\rho) + z_n(t,\rho) - a\xi'_{jz_{n+1}}(t,\rho) \right\},$$

where the integral on the right converges absolutely and uniformly with respect to x, t in the domain $0 \le x, t < \infty$.

Denote by $\widehat{G}_A$ the totality of all vector-functions $g(x)$ satisfying the following conditions: 1) $g(x)$, $l(g)$ are summable on the interval $[0,\infty)$; 2) $g^{(\nu)}(x)$, $\nu = 1,\dots,2n-1$, exist and are absolutely continuous on every finite interval $[0,b]$; 3) $u_\nu(g) = 0$, $\nu = 1,2,\dots,n$. Then, with the aid of formula (21), one can obtain an expansion of the function $g(x)$ in the eigenfunctions of the operator L_A and an analogue of Parseval' s equality.

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Moscow State University
named after M. V. Lomonosov

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Note: Figure translations are in progress. See original paper for figures.