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Abstract

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ON A GENERALIZATION OF T. REGGE' S THEOREM

(Presented by Academician N. N. Bogolyubov on 14 VI 1962)

Among the interesting results of T. Regge ([2]), devoted to the study of the analytic properties of the scattering amplitude on a potential, there is the following

Theorem 1 (Regge). Let $f(z)$ be a function regular on the interval $[-1, 1]$ and such that its expansion in Legendre polynomials

$$f(z) = \sum_{n=1}^{\infty} a_n P_n(z) \tag{1}$$

converges inside some ellipse with foci at the points $-1, 1$. Let there exist a function $\varphi(l)$, analytic and regular in the half-plane $\operatorname{Re} l \geqslant l_0$, bounded as $l \to \infty$ in any direction in the half-plane $\operatorname{Re} l \geqslant l_0$, and let the relations

$$a_n = \varphi(n), \qquad n > l_0.$$

be satisfied. Then the function $f(z)$ is analytic and regular in the entire complex z-plane, except for the cut $x_0 < \operatorname{Re} z < \infty$, where x_0 is a point situated on the ellipse of convergence of the expansion (1).

Regge proves the theorem formulated above by means of the Watson transform.

In the present note we shall show that, with the aid of one result of Faber ([3,4]), the theorem stated above can be obtained as a consequence of the theorems of Le Roy–Lindelöf, proved at the beginning of this century. The results of the authors mentioned and one theorem of Carlson (([6]), p.56) also open the way to certain generalizations of Regge' s theorem, which may be of interest for applications if one takes into account that the properties of the interpolating function $\varphi(l)$ are dictated by the analytic properties of the potential ([2]).

Theorem 2 (Le Roy–Lindelöf). Let the analytic function $\varphi(l)$ of the variable $l = \tau + it$ satisfy the conditions: 1) $\varphi(l)$ is regular in some half-plane $\tau \geqslant a$; 2) there exists a number $\vartheta < \pi$ such that, for arbitrary $\varepsilon > 0$ and sufficiently large $\rho > 0$, the inequality

$$|\varphi(\alpha + \rho e^{i\psi})| < e^{(\vartheta+\varepsilon)\rho}$$

holds for $-\pi/2 \leq \psi \leq \pi/2$.

Under these conditions the function $F(x)$ of the variable $x = re^{i\theta}$, defined by the power series

$$F(x) = \varphi(0) + \varphi(1)x + \varphi(2)x^2 + \cdots + \varphi(n)x^n + \cdots, \tag{2}$$

is holomorphic for all x lying inside the angle $\vartheta < \theta < 2\pi - \vartheta$. In particular, if $\vartheta = 0$, then $F(x)$ is holomorphic in the whole plane, except for the segment $1 - \infty$ of the real axis.

Regge' s result is a consequence of the Le Roy—Lindelöf theorem, if one takes into account Faber' s theorem (see ([3]), p. 92), according to which

between the singular points α of the function $F(x)$ (2) and the singular points β of the function $f(z)$ (1), in view of the relations $a_n = \varphi(n)$, there is the dependence

$$\beta = \frac{1}{2}\left(\alpha + \frac{1}{\alpha}\right). \tag{3}$$

It is clear that the transformation (3) carries the positive half-plane of the x-plane into the positive half-plane of the z-plane.

Faber' s theorem also makes it easy to formulate a generalization of Regge' s theorem, imitating the above-mentioned Le Roy-Lindelöf theorem. We also point to Carlson' s theorem (([6], p. 56)), with the aid of which one can formulate an analogous theorem for expansions in Legendre polynomials.

In connection with the application of Regge' s theorem to relativistic theories ([7,8]), the problem arose ([8]) of the uniqueness of the interpolating function $\varphi(l)$. We note that, along with the Blaschke theorem used by V. I. Gribov, in a more general formulation of the problem one may use Carlson' s uniqueness theorem (([6], p. 64)). Theorems on the asymptotic behavior of the function (1) can also be easily obtained from Lindelöf' s theorems (([5], p. 113)).

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CITED LITERATURE

[1] T. Regge, Nuovo Cim., **14**, No. 5, 951 (1959).
[2] A. Bottino, A. M. Longoni, T. Regge, Nuovo Cim., **23**, No. 6, 954 (1962).
[3] P. Montel, *Leçons sur les séries de polynomes à une variables*, Paris, 1910, p. 92.
[4] O. S. Parasyuk, DAN, **145**, No. 6 (1962).
[5] E. Lindelöf, *Le calcul des résidus et ses applications à la théorie des fonctions*, Paris, 1905.
[6] E. Hille, *Analytic Function Theory*, 2, Boston–N. Y., 1962.
[7] V. N. Gribov, ZhETF, **41**, issue 6, 1962 (1961).
[8] V. N. Gribov, ZhETF, **42**, issue 5, 1260 (1962).

Note: Figure translations are in progress. See original paper for figures.