ON ONE METHOD FOR SOLVING THE POISSON EQUATION

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Abstract

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ON ONE METHOD FOR SOLVING THE POISSON EQUATION

(Presented by Academician S. L. Sobolev, 27 X 1961)

To find the solution of the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = F(x, y),$$

satisfying the boundary condition

$$u\big|_S = \varphi(s),$$

we consider the usual 5-point difference approximation

$$\Delta^2_{xx} v_{ij} + \Delta^2_{yy} v_{ij} = F_{ij} \quad \text{for } (i, j) \in \Omega_h; \tag{1}$$

$$v_{ij} = \varphi_{ij} \quad \text{for } (i, j) \in S_h; \tag{2}$$

$$\Delta^2_{xx} v_{ij} = \frac{v_{i+1,j} - 2v_{ij} + v_{i-1,j}}{h^2}, \qquad \Delta^2_{yy} v_{ij} = \frac{v_{i,j+1} - 2v_{ij} + v_{i,j-1}}{h^2}.$$

There exist numerous iterative methods for solving (1), (2), among which the methods of ([1,2]), in the case of rectangular domains, have the greatest rate of convergence.

In the present note an iterative method is proposed for solving (1), (2), which for rectangular domains has the same rate of convergence as the methods of ([1,2]), i.e., finding the solution of (1), (2) with accuracy ε requires $\succ \frac{\ln h \, \ln \varepsilon}{h^2}$ arithmetic operations. For domains composed of a finite number of rectangles with sides parallel to the coordinate axes, one can find the solution of (1), (2) with accuracy ε, expending $\succ \frac{\ln^2 h \, |\ln \varepsilon|}{h^2}$ arithmetic operations, if the methods of ([1,2]) or the proposed method are combined with the difference analogue of Schwarz' s method.

1. Let $\bar{\Omega}$ be a rectangle: $0 \le x \le a,\ 0 \le y \le b$; h is the mesh step; $u(ih, jh) = u_{ij}$;

$$\Omega_h\{(i,j):\ i = 1, 2, \dots, N_1 - 1;\ j = 1, 2, \dots, N_2 - 1\};$$

$$S_h\{(i,j):\ i = 0, N_1 \text{ or } j = 0, N_2\}; \qquad \|u\|_{L_2} = \left(h^2 \sum_{ij} u_{ij}^2\right)^{1/2}.$$

Expending $\succ \dfrac{1}{h^2}$ arithmetic operations, problem (1), (2) is easily reduced to the problem

$$\Delta_{xx}^2 u_{ij} + \Delta_{yy}^2 u_{ij} = f_{ij} \quad \text{for } (i,j) \in \Omega_h; \tag{3}$$

$$u_{ij} = 0 \quad \text{for } (i,j) \in S_h. \tag{4}$$

The following iterative method is proposed for solving problem (3), (4). Suppose $u_{ij}^{(n)}$ is given—the n-th approximation for u_{ij}. Then $u_{ij}^{(n+1)}$ will be determined from the system

$$\left(E - \tau_n \Delta_{xx}^2\right) u_{ij}^{(n+1/2)} = \left(E + \tau_n \Delta_{xx}^2\right) u_{ij}^{(n)}; \tag{5}$$

$$\left(E - \tau_n \Delta_{yy}^2\right) u_{ij}^{(n+1)} = \left(E + \tau_n \Delta_{yy}^2\right) u_{ij}^{(n+1/2)} + \tau_n \tilde{f}_{ij}^{(n)}, \tag{5'}$$

$$(i,j) \in \Omega_h,$$

where $u_{ij}^{(n)}$, $u_{ij}^{(n+1/2)}$, $u_{ij}^{(n+1)}$ satisfy (4); $\tau_n > 0$ is an iteration parameter; E is the identity operator:

$$\tilde{f}_{ij}^{(n)} = \begin{cases} 0, & \text{for } (i,j) \in S_h, \\ (E - \tau_n \Delta_{xx}^2)^{-1} f_{ij}, & \text{for } (i,j) \in \Omega_h. \end{cases} \tag{6}$$

It is not difficult to verify that the transition by this method from $u^{(n)}$ to $u^{(n+1)}$ requires $\asymp \dfrac{1}{h^2}$ operations.

By virtue of (4), (6), eliminating $u^{(n+1/2)}$, we obtain

$$(E - \tau_n \Delta_{xx}^2)(E - \tau_n \Delta_{yy}^2) u_{ij}^{(n+1)} = (E + \tau_n \Delta_{xx}^2)(E + \tau_n \Delta_{yy}^2) u_{ij}^{(n)} + f_{ij}.$$

Therefore, for the error $e_{ij}^{(n)} = u_{ij} - u_{ij}^{(n)}$, we have

$$(E - \tau_n \Delta_{xx}^2)(E - \tau_n \Delta_{yy}^2) e_{ij}^{(n+1)} = (E + \tau_n \Delta_{xx}^2)(E + \tau_n \Delta_{yy}^2) e_{ij}^{(n)}. \tag{7}$$

Let

$$v_{kl}(i,j) = \sin \frac{k\pi ih}{N_1} \sin \frac{l\pi jh}{N_2}; \qquad \Delta_{xx} v_{kl} = -\lambda_k^2 v_{kl}; \qquad \Delta_{yy} v_{kl} = -\mu_l^2 v_{kl}.$$

Considering the expansions $e^{(n+1)} = \sum_{k,l} a_{kl}^{(n+1)} v_{kl}$, it is not difficult to obtain $a_{kl}^{(n+1)} = \rho^{(n+1)}(k,l) a_{kl}^{(0)}$, where

$$\rho^{(n+1)}(k,l) = \frac{\prod_{s=0}^{n} (1 - \tau_s \lambda_k^2)(1 - \tau_s \mu_l^2)}{\prod_{s=0}^{n} (1 + \tau_s \lambda_k^2)(1 + \tau_s \mu_l^2)}.$$

Obviously, $m \le \lambda_k^2 \le 4/h^2$, $m \le \mu_l^2 \le 4/h^2$, where $m > 0$ and does not depend on h. Take any $r > 1$ and form the following set of τ_n:

$$\tau_0 = \frac{r}{m}, \qquad \tau_{n+1} = \frac{\tau_n}{r} \quad (n = 0, 1, \dots, M-1), \tag{8}$$

where M is the smallest integer for which $\tau_M \frac{4}{h^2} \le \frac{1}{r}$. It is easy to verify that $M \asymp |\ln h|$.

Lemma 1. If $M+1$ iterations have been performed by method (5), (5′) with the set of parameters (8), then for all k, l the inequality

$$|\rho^M(k,l)| < \left(\frac{r-1}{r+1} \prod_{k=1}^{M} \left(\frac{r^k - 1}{r^k + 1} \right) \right)^2.$$

holds.

We note that method (^{1}) also leads to relation (7); therefore Lemma 1 is a certain strengthening of the estimates from (^{1}).

With the aid of this lemma the following theorem is proved:

Theorem 1. To find the solution of (3), (4) with accuracy ε in the sense of the norm in L_2, it is sufficient to repeat a cycle of $M+1$ iterations with parameters (8) $\asymp |\ln \varepsilon|$ times; in this case the total number of arithmetic operations performed will be

$$\asymp \frac{\ln\varepsilon \ln h}{h^2}.$$

If one considers method (5), (5′) with constant τ_n, equal to τ, then the following theorem can be established:

Theorem 2. To find the solution of (3), (4) with accuracy ε, it is sufficient to perform $\asymp \dfrac{|\ln\varepsilon|}{h}$ iterations by method (5), (5′) with $\tau \asymp h$; the number of arithmetic operations performed in such a method will be $\asymp \dfrac{|\ln\varepsilon|}{h^3}$.

We note that the same asymptotics for the number of arithmetic operations is obtained also for methods ([1], [2]) with one iteration parameter.

2. Let us now consider the case when $\overline{\Omega} = \bigcup_{k=1}^{p} \overline{D}_k$, where $\overline{D}_k$ is a rectangle with boundary S_k, composed of line segments parallel to the coordinate axes. For simplicity of exposition we restrict ourselves to the case $p = 2$. Denote by $\overline{D}_k^h$ the set of grid points belonging to D_k, and denote the boundary points of $\overline{D}_k^h$ by S_k^h. Introduce also the following notation: $D_k^h = \overline{D}_k^h \setminus S_k^h$, $\Omega_h = D_1^h \cup D_2^h$; $\overline{\Omega}_h = \overline{D}_1^h \cup \overline{D}_2^h$; $\alpha_k = S_k^h \cap D_l^h$ $(l \neq k)$; $\alpha_k' = S_k^h \setminus \alpha_k$.

There is the following lemma, which is a difference analogue of a particular case of Schwarz’ s lemma ([3]):

Lemma 2. If the function $\omega^{(k)}$ satisfies the difference Laplace equation

$$\Delta_h \omega_{ij}^{(k)} = (\Delta_{xx}^2 + \Delta_{yy}^2)\omega_{ij}^{(k)} = 0$$

for $(i,j) \in D_k^h$ and the boundary condition

$$\omega_{ij}^{(k)} = \begin{cases} 0 & \text{for } (i,j) \in \alpha_k, \\ 1 & \text{for } (i,j) \in \alpha_k', \end{cases}$$

then $0 \le \omega_{ij}^{(k)} \le q < 1$, for $(i,j) \in \alpha_l$, where the constant q does not depend on h; $k = 1, 2;$ $l = 2, 1$.

Since problem (1), (2), with an expenditure of arithmetic operations $\asymp \frac{1}{h^2}$, is reduced to the problem

$$\Delta_h \omega_{ij} = 0 \quad \text{for } (i,j) \in \Omega_h; \tag{9}$$

$$\omega_{ij} = \varphi_{ij} \quad \text{for } (i,j) \in S_h, \tag{10}$$

instead of solving (1), (2) we shall solve (9), (10).

Consider the following iterative process, combining the difference analogue of Schwarz’ s method and any one of the methods ([1],[2]), (5), (5′). Suppose that in $\overline{D}_1^h$ and $\overline{D}_2^h$ functions $\overline{u}^{(0)}$, $\overline{v}^{(0)}$ are given, satisfying (10) and coinciding on $\overline{D}_1^h \cap \overline{D}_2^h$.

We seek the function $u^{(1)}$ from the conditions

$$\Delta_h u_{ij}^{(1)} = 0 \quad \text{for } (i,j) \in D_1^h;$$

$$u_{ij}^{(1)} = \begin{cases} \varphi_{ij} & \text{for } (i,j) \in \alpha_1, \\ \overline{v}_{ij}^{(0)} & \text{for } (i,j) \in \alpha_1'. \end{cases}$$

For this purpose, as the zeroth approximation for $u^{(1)}$ we take the function

$$\tilde{u}^{(1)} = \begin{cases} \overline{u}_{ij}^{(0)} & \text{for } (i,j) \in \overline{D}_1^h \setminus \alpha_1', \\ \overline{v}_{ij}^{(0)} & \text{for } (i,j) \in \alpha_1', \end{cases}$$

and perform L cycles of iteration by one of the methods (^{1},^{2}), (5), (5′). As a result we obtain $\overline{u}^{(1)}$.

We seek $v^{(1)}$ from the conditions:

$$\Delta_h v_{ij}^{(1)} = 0 \quad \text{for } (i,j) \in D_2^h;$$

$$v_{ij}^{(1)} = \begin{cases} \varphi_{ij} & \text{for } (i,j) \in \alpha_2, \\ \overline{u}_{ij}^{(1)} & \text{for } (i,j) \in \alpha_2'. \end{cases}$$

As the zeroth approximation for $v^{(1)}$ we take

$$\tilde{v}^{(1)} = \begin{cases} \overline{v}_{ij}^{(0)} & \text{for } (i,j) \in \overline{D}_2^h \setminus \alpha_2, \\ \overline{u}_{ij}^{(1)} & \text{for } (i,j) \in \alpha_2', \end{cases}$$

and then L cycles of iteration are performed. We obtain $v^{(1)}$. From $v^{(1)}$ we construct $u^{(2)}$, and so on.

Theorem 3. If $L \asymp |\ln h|$, then as $n \to \infty$ the functions $\overline{u}^{(n)}$, $\overline{v}^{(n)}$ tend to ω, and, to obtain the estimate,

$$\max\{\|\omega - \overline{u}^{(n)}\|_c,\ \|\omega - \overline{v}^{(n)}\|_c\} < \varepsilon \max\{\|\omega - \overline{u}^{(0)}\|_c,\ \|\omega - \overline{v}^{(0)}\|_c\}$$

it suffices to take $\asymp |\ln \varepsilon|$. The number of arithmetic operations performed in this case will be

$$\asymp \frac{|\ln \varepsilon|\, \ln^2 h}{h^2}.$$

The proof of this theorem is carried out in approximately the same way as the proof of Theorem 3 in paper (4).

Remark 1. The methods ([1],[2]), (5), (5′) carry over to the case of the first boundary-value problem in a rectangle for the equation $\mathcal{L}u = f$, where $\mathcal{L}$ is a positive-definite self-adjoint elliptic operator of order $2m$ with separable variables.

Remark 2. In the case of s variables, where $s \geqslant 3$, the method (5), (5′), as well as the method ([1]), does not carry over; therefore Theorem 3 remains valid only for the method ([2]).

Remark 3. A certain generalization of the method (5), (5′) is possible. Namely, instead of (5), (5′) one considers the system of equations

$$\begin{gathered}\left(E-\sigma_1\tau_n\Delta^2_{xx}\right)u_{ij}^{(n+1/2)} = \left(E+(1-\sigma_1)\tau_n\Delta^2_{xx}\right)u_{ij}^{(n)},\\ \left(E-\sigma_2\tau_n\Delta^2_{yy}\right)u_{ij}^{(n+1)} = \left(E+(1-\sigma_2)\tau_n\Delta^2_{yy}\right)u_{ij}^{(n+1/2)} + \tilde{f}_{ij}^{(n)},\\ \tilde{f}_{ij}^{(n)} = \left(E-\sigma_1\tau_n\Delta^2_{xx}\right)^{-1} f_{ij}.\end{gathered}$$

If $1/2 \leqslant \sigma_i \leqslant 1$ $(i = 1, 2)$, then Theorems 1, 2, 3 are also valid for this generalization (5), (5′).

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