MATHEMATICS

Corresponding Member of the Academy of Sciences of the USSR
Yu. V. LINNIK, B. F. SKUBENKO

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Abstract

Full Text

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ON THE ASYMPTOTICS OF INTEGER MATRICES OF THE THIRD ORDER

Questions of the ergodic theory of integer matrices, developed by the authors in papers ([1],[2]), require, in particular, the solution of the following problem. Consider integer square matrices (i.e., matrices) X with prescribed determinant N. Consider the surface $\det(X) = N$ and single out on it some "reasonably defined" domain Ω. For large N it is required to specify the asymptotics for the number of integer matrices lying in the domain Ω.

This problem was solved for matrices of the second order in ([3]). In the present note we prove its solution for matrices of the third order. Apparently, the corresponding generalization of the method indicated here will lead to a solution for matrices of arbitrary order.

Let a domain Ω be given among unimodular matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

with boundary consisting of a fixed number of smooth surfaces and with the condition $|a_{ij}| < K$ (K is an arbitrarily large constant). Further, let $N > 5$ be any natural number. Introduce the substitution $x_{ij} = a_{ij}N^{1/3}$. We have, obviously,

$$N = \begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix}. \quad (1)$$

Denote by $f(\Omega, N)$ the number of integer primitive solutions of equation (1) with the condition $A \in \Omega$, $x_{ij} = a_{ij} \cdot N^{1/3}$, $A = (a_{ij})$.

Theorem.

$$f(\Omega, N) \sim G(N)\frac{\text{mes}(\Omega)}{\zeta(2)\zeta(3)}$$

for fixed Ω and as $N \to \infty$, where

$$G(N) = \prod_{p_i^{n_i}} \frac{(p_i^{n_i+2}-1)(p_i^{n_i+1}-1)-(p_i^{n_i-1}-1)(p_i^{n_i-2}-1)}{(p_i-1)^2(p_i+1)};$$

$$N = p_1^{n_1} p_2^{n_2} \cdots p_i^{n_i} \cdots p_k^{n_k}$$

is the canonical decomposition of N; $\mathrm{mes}(\Omega)$ is the Haar measure on the group of unimodular matrices.

This theorem follows directly from a lemma, which is of independent interest.

Lemma. For any given integers $a_{11}, a_{12}, a_{21}, a_{22}$ satisfying $|a_{11}|$, $|a_{12}|$, $|a_{21}|$, $|a_{22}| < N^{1/3}$, i.e. $(a_{11}, a_{12}, a_{21}, a_{22}) = 1$,

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = q < N^{2/3},$$

there exist primitive solutions of the equation

$$\begin{vmatrix} a_{11} & a_{12} & v_1 \\ a_{21} & a_{22} & u_1 \\ v_2 & u_2 & t \end{vmatrix} = N. \tag{2}$$

in the region

$$n_i < v_i \le m_i, \qquad k_i < u_i \le l_i,$$

$$|n_i| \asymp |m_i| \asymp |k_i| \asymp |l_i| \asymp N^{1/3},$$

$$m_1 - n_1 = \Delta_1, \qquad m_2 - n_2 = \Delta_2, \qquad l_1 - k_1 = \Delta_3, \qquad l_2 - k_2 = \Delta_4, \qquad \Delta_j \asymp N^{1/3},$$

and the number of such solutions is expressed by the formula

$$W(\Delta_1, \Delta_2, \Delta_3, \Delta_4, q, N) = \prod_{j=1}^{4} \Delta_j \, \frac{\varphi(q)}{q^2} \left(1 + O\left(q^{-1/13}\right)\right),$$

where

$$q = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

From this lemma and Lemma 15 of the paper ([3]), the present theorem is derived. Our lemma is proved with the aid of the known lemma of I. M. Vinogradov on Fourier series ([4]) and the following observation.

Let

$$\begin{vmatrix} a_{11} & a_{12} & a_2 \\ a_{21} & a_{22} & b_2 \\ a_1 & b_1 & \sigma \end{vmatrix} = N$$

be an arbitrary integral primitive solution of equation (2) with the condition g.c.d. $(a_{11}, q) = 1$; then all other solutions of equation (2) will have the form (in self-explanatory notation)

$$\begin{vmatrix} a_{11} & a_{12} & a_2 z' + y a_{11} \\ a_{21} & a_{22} & b_2 z' + y a_{21} \\ a_1 z + x a_{11} & b_1 z + x a_{12} & t \end{vmatrix} = N,$$

where $zz' \equiv 1 \pmod q$, and t is determined uniquely by N and all the other components.

To prove the lemma it is necessary to estimate from above the sum

$$\sum_{\substack{l_1, l_2, l_3, l_4 = -\infty \\ |l_1|+|l_2|+|l_3|+|l_4| \neq 0}}^{+\infty} C_{l_1 l_2 l_3 l_4} \sum_{\substack{x, y, z \pmod q \\ (z,q)=1}} \exp\left[\frac{2\pi i}{q} \{z(a_1 l_1 + b_2 l_2) + z'(a_2 l_3 + b_2 l_4) + \right.$$

$$\left. + x(a_{11} l_1 + a_{12} l_2) + y(a_{11} l_3 + a_{12} l_4)\}\right];$$

$$1)\quad |C_{l_1 l_2 l_3 l_4}| \leq \sum_{j=1}^{4} \Delta_j q^{-4}; \qquad 2)\quad |C_{l_1 l_2 l_3 l_4}| \leq \frac{1}{(\lambda_1 \lambda_2 \lambda_3 \lambda_4)^r},$$

where

$$\lambda_j \geq \max\left(\Delta_j^{-1} q,\ \frac{1}{r} \left|\Delta_j l_j q^{-1} q^{-1/2+\varepsilon}\right|\right)$$

(ε arbitrarily small).

The written sum is estimated as $O(q^{3-1/12+\varepsilon_1})$ (ε_1 is an arbitrarily small positive quantity) with the aid of the known estimates of André Weil ([5]).

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.