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Abstract

Full Text

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CONGRUENCES OF PARABOLAS IN EQUIAFFINE GEOMETRY

(Presented by Academician P. S. Novikov, 14 IV 1962)

In the present work a nondegenerate congruence of parabolas (a congruence P) is considered, i.e., it is assumed that at least one focal surface of the congruence: 1) is not the envelope of a family of planes of the parabolas; 2) is not a torsal surface; 3) has non-asymptotic focal lines.

§ 1. **The canonical frame of a congruence P . Foci and focal families.** Place the origin A of the frame $\{A; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ at the focus that forms a nondevelopable focal surface with non-asymptotic focal lines. The vector \mathbf{e}_1 is directed along the tangent to the parabola at the point A , the vector \mathbf{e}_3 —along the diameter of the parabola passing through the point A ; finally, the vector \mathbf{e}_2 is directed along the tangent to the line conjugate to the focal one. With a corresponding normalization of the vectors \mathbf{e}_1 and \mathbf{e}_2 , the derivation formulas of the canonical frame have the form:

$$\begin{aligned}
 dA &= \omega^1 \mathbf{e}_1 + \omega^2 \mathbf{e}_2, \\
 d\mathbf{e}_1 &= -\frac{1}{8}\{(3a+c)\omega^1 + (3b+e)\omega^2\}\mathbf{e}_1 + (f\omega^1 + g\omega^2)\mathbf{e}_2 + \omega^1 \mathbf{e}_3, \\
 d\mathbf{e}_2 &= \{(f-b)\omega^1 + (g-c)\omega^2\}\mathbf{e}_1 + \frac{1}{8}\{(a+3c)\omega^1 \\
 &\quad + (b+3e)\omega^2\}\mathbf{e}_2 - \omega^2 \mathbf{e}_3, \\
 d\mathbf{e}_3 &= (h\omega^1 + k\omega^2)\mathbf{e}_1 + (r\omega^1 + s\omega^2)\mathbf{e}_2 \\
 &\quad + \frac{1}{4}\{(a-c)\omega^1 + (b-e)\omega^2\}\mathbf{e}_3.
 \end{aligned} \tag{1}$$

The invariants $a, b, c, e, f, g, h, k, r, s$ satisfy a system of six exterior quadratic equations obtained from the structure equations of affine space.

The equation of the parabola with respect to the canonical frame has the form

$$(x^1)^2 - 2px^3 = 0, \quad x^2 = 0, \tag{2}$$

where $p \neq 0$ is an invariant of the congruence. It turns out that a congruence P exists and is determined with arbitrariness of five functions of two arguments.

The focal points and focal families of a congruence P are determined from equations (2) and the equations

$$(x^1)^2\omega_1^1 + x^1x^3\omega_3^1 + (1-p)\omega^1x^1 + (dp - p\omega_3^3)x^3 = 0, \quad (3)$$

$$x^1\omega_1^2 + x^3\omega_3^2 + \omega^2 = 0.$$

It follows from this that if the diameters of the parabolas of the congruence P do not form a bundle of parallel straight lines, then the congruence P has five focal families and five focal surfaces, among which some may coincide.

§ 2. Associated geometric images.

To every congruence P the following geometric images are invariantly associated:

- 1) The congruences t_ε -indicatrices of the focal surface

$$(x^1)^2 + \varepsilon(x^2)^2 = p^2, \quad x^3 = 0, \quad \varepsilon = \pm 1. \quad (4)$$

The congruence t_{-1} -indicatrix is a congruence of indicatrices in the sense of N. G. Tuganov ⁽¹⁾, and its focal families

$$\omega_1^1(\omega^2)^2 - \omega_2^2(\omega^1)^2 + \omega^1\omega^2(\omega_1^2 - \omega_2^1) + [(\omega^1)^2 - (\omega^2)^2] d \ln p = 0 \quad (5)$$

are the lines of centers ⁽¹⁾ of the focal surface (A) .

- 2) The rectilinear congruence of diameters passing through the points of tangency of the parabolas with the focal surface (A) .
- 3) The rectilinear congruences of focal tangents $\{A, \mathbf{e}_1\}$ and conjugate tangents $\{A, \mathbf{e}_2\}$.

If $g(b-f) \neq 0$, then the second foci F_1, F_2 of these congruences are determined by the formulas

$$\mathbf{F}_1 = A - \frac{1}{g}\mathbf{e}_1, \quad \mathbf{F}_2 = A + \frac{1}{b-f}\mathbf{e}_2. \quad (6)$$

- 4) The envelopes (M_1) and (M_2) of the coordinate planes $\{\mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{e}_1, \mathbf{e}_3\}$.
- 5) The rectilinear congruence of affine normals of the focal surface. The vector \mathbf{e} of the affine normal is determined by the formula

$$\mathbf{e} = \mathbf{e}_3 + \frac{1}{4}\{(C-a)\mathbf{e}_1 + (b-e)\mathbf{e}_2\}. \quad (7)$$

§ 3. Some classes of congruences of parabolas.

- 1) Congruences all diameters of whose parabolas form a bundle of parallel straight lines (congruences P_0). The congruences P_0 are characterized by the relations $h = k = r = s = 0$, are determined with arbitrariness of three functions of two arguments, and in the general case have three focal surfaces and three focal families.

For $f = g = a = b = 0$, $p = 1$, with arbitrariness of two functions of one argument, there is distinguished a subclass of these congruences having the surface (A) as their only focal surface (congruences P'_0).

The lines $\omega^1 = 0$ and $\omega^2 = 0$ of the focal surface of a congruence P'_0 are shadow lines (cylindrical lines), and the focal lines $\omega^2 = 0$ are planar. The focal surface of a congruence P'_0 is an affine surface of translation ((²), p. 75).

If the lines $\omega^1 = 0$ are also planar, then, with arbitrariness of one function of one argument, we obtain a subclass of congruences P''_0 (congruences P''_0). The unique invariant e different from zero of the congruences P''_0 is constant along the focal line. From equation (5) we conclude that the focal lines $\omega^2 = 0$ are lines of centers.

- 2) Congruences having one doubled focal surface (A) (congruences P_1). They are characterized by the equality $p = 1$ and are determined with arbitrariness of four functions of two arguments. Of interest is the selection of those congruences P_1 for which the diameters of the parabolas, associated with the focal surface (A) , are affine normals of the surface (A) (congruences P'_1). The congruences P'_1 are characterized by the equalities $c = a$, $e = b$, $r + k = 0$ and are determined with arbitrariness of two functions of two arguments. Their canonical frame is the Shcherbakov C -frame of the focal surface (A) ((²), p. 58). The two families of lines of centers of the focal surface (A) of a congruence P'_1 are asymptotic lines of the surface (A) .

Some subclasses of congruences P'_0 have been considered.

- a) Congruences $a = b \neq 0$. They are determined with the arbitrariness of one function of two arguments. All three families of lines of the centers of the focal surface of such a congruence are asymptotic (one family is doubled).
- b) Congruences $g(b - f) = 0$. They are determined with the arbitrariness of one function of two arguments and are characterized by the fact that the focal lines of the surface (A) or the lines conjugate to them are lines of shadow. If simultaneously $g = 0$, $b - f = 0$, then the focal surface (A) is a translation surface. A congruence P'_0 possessing this property is determined with the arbitrariness of four functions of one argument.
- c) Congruences $a = b = 0$ are determined with the arbitrariness of one function of two arguments. The focal surface (A) of such congruences is

a quadric.

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¹ N. G. Tuganov, DAN, **94**, No. 2, 189 (1954).

² R. N. Shcherbakov. *A Course in Affine and Projective Differential Geometry*, Tomsk, 1960.

Note: Figure translations are in progress. See original paper for figures.

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