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Abstract

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NUTATIONAL OSCILLATIONS AND DRIFT OF A SYNCHRONOUS GYROSCOPE MOUNTED IN A GIMBAL SUSPENSION

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A gyroscope rotating with a high angular velocity about its own axis must experience the braking action of the friction torque in the bearings and the torque due to resistance of the external medium. To balance these torques, the gyroscope is supplied with an electric drive, so that the inner ring and the gyroscope constitute the stator and rotor of a synchronous or asynchronous motor (gyromotor). In a synchronous gyroscope, constancy of the angular velocity of its proper rotation is ensured by pulling the gyroscope into synchronism with the rotating magnetic field of the stator.

If the rings of the gimbal suspension are not perpendicular, then the torque of the forces of the stator magnetic field influences the nutational oscillations and drift of the gyroscope.

1. Equations of motion of a synchronous gyroscope. We shall compile the equations of motion of a synchronous gyroscope by Lagrange' s method. To determine the position of the gyroscope in inertial space we use reference systems attached to the separate bodies, and the generalized coordinates of the system α, β, γ, introduced in work ([1]). We shall assume the generalized forces Q_α and Q_β to be equal to zero. The generalized force Q_γ consists of the constant resistance torque M and the torque of the forces of the stator magnetic field, which depends on the angle of rotation of the rotor relative to the rotating field of the stator:

$$Q_\gamma = -M - f(\theta)_0$$

Since in the present work the problem is posed of studying the nutational oscillations of the gyroscope and its drift relative to the axis of the outer suspension ring, for simplicity we put $f(\theta) = k\theta$. We borrow the expression for the kinetic energy from work ([1]). After writing the Lagrange equations, we make in them the substitution

$$\gamma = \omega t + \theta,$$

where ω is the angular velocity of rotation of the magnetic field. Taking into account the remarks made, we write the general equations of motion of the synchronous gyroscope in the form

$$[A_2 + (A + A_1)\cos^2\beta + (C + C_1)\sin^2\beta]\ddot{\alpha} - 2(A + A_1 -$$

$$-C - C_1)\dot{\alpha}\dot{\beta}\cos\beta\sin\beta + H\dot{\beta}\cos\beta + C(\ddot{\theta}\sin\beta + \dot{\theta}\dot{\beta}\cos\beta) = 0,$$

$$(A + B_1)\ddot{\beta} + (A + A_1 - C - C_1)\dot{\alpha}^2\cos\beta\sin\beta - H\dot{\alpha}\cos\beta - C\dot{\theta}\dot{\alpha}\cos\beta = 0,$$

$$C(\ddot{\theta} + \dot{\alpha}\dot{\beta}\cos\beta + \ddot{\alpha}\sin\beta) + k(\theta + \theta_0) = 0,$$

here $H = C\omega$ and $\theta_0 = M/k$. These equations of motion have two first integrals

$$[A_2 + (A + A_1)\cos^2\beta + (C + C_1)\sin^2\beta]\dot{\alpha} + H\sin\beta + C\dot{\theta}\sin\beta = K,$$

$$[A_2 + (A + A_1)\cos^2\beta + (C + C_1)\sin^2\beta]\dot{\alpha}^2 + (A + B_1)\dot{\beta}^2 + C\dot{\theta}^2 -$$

$$-2C\dot{\theta}\dot{\alpha}\sin\beta + V(\theta) = \text{const},$$

where $V(\theta) = k(\theta + \theta_0)^2$.

The first of the integrals shows that the projection of the angular momentum of the system onto the axis of the outer ring remains constant. The second integral is analogous to the energy integral.

The equations of motion of a synchronous gyroscope admit the particular solution

$$\alpha = \alpha_0, \qquad \beta = \beta_0, \qquad \theta = -\theta_0,$$

corresponding to stationary rotation of the gyroscope relative to an axis fixed in inertial space. The first integrals given above can be used to prove the stability of the stationary rotation with respect to the generalized coordinates θ, β and the velocities $\dot{\alpha}, \dot{\beta}, \dot{\theta}$. To investigate the stability of the stationary rotation with respect to the cyclic coordinate α, we shall subsequently need an expression for the moment of the reactions of the supports of the outer ring. Determining this moment by the method of A. I. Lur' e (^{2}), we find that

$$P_{y_2} = 0, \qquad P_{z_2} = (A_2 + C_1)\ddot{\alpha}\ \mathrm{tg}\,\beta + (C_1 + B_1 - A_1)\dot{\alpha}\dot{\beta} + \frac{Q_\gamma}{\cos\beta};$$

here P_{y_2}, P_{z_2} are the projections of the required moment on the coordinate axes $Ox_2y_2z_2$, associated with the outer ring.

2. Nutational oscillations and drift of a synchronous gyroscope. We shall now study small oscillations of a synchronous gyroscope and its stability with respect to the cyclic coordinate α. At the initial instant, the outer ring is impulsively imparted the angular velocity Ω. We write the equations of the perturbed motion in the form

$$\alpha = \alpha_0 + ut + \xi(t), \qquad \beta = \beta_0 + \eta(t), \qquad \theta = -\theta_0 + \zeta(t),$$

where u is the angular velocity of precession of the outer ring, of small order Ω^2/ω. The functions ξ, η, ζ describe small oscillations of the gyroscope relative to the reference frame $OXYZ$, rotating with constant angular velocity about the axis OX, which coincides with the axis of the outer ring of the gimbal suspension. Substituting these expressions into the general equations of motion and neglecting small quantities of second order, we obtain the following system of linear equations for ξ, η, ζ:

$$J_0\ddot{\xi} + H\cos\beta_0\dot{\eta} + C\sin\beta_0\ddot{\zeta} = 0,$$

$$I_1\ddot{\eta} - H\cos\beta_0\dot{\xi} = 0,$$

$$C(\ddot{\zeta} + \ddot{\xi}\sin\beta_0) + k\zeta = 0,$$

$$J_0 = A_2 + (A + A_1)\cos^2\beta_0 + (C + C_1)\sin^2\beta_0,$$

$$I_1 = A + B_1.$$

Introduce the notation

$$\nu_0 = H\cos\beta_0/\sqrt{I_0 I_1},$$

$$\nu = H\cos\beta_0/\sqrt{J_0 I_1}, \qquad \lambda = \sqrt{k/C};$$

Fig. 1. $\mathrm{tg}\,\angle aO\lambda = J_0/I_0, \quad \angle bO\lambda = 45^\circ$

Fig. 1.

Figure 1: Fig. 1.

here ν_0 and ν are the frequencies of the nutational oscillations of the gyroscope without friction on its own-rotation axis and without a drive (we shall call such a gyroscope ideal) and of a gyroscope with servocoupling $d\gamma/dt = \omega$, respectively; λ is the frequency of small oscillations of the gyroscope relative to the stator magnetic field in the case when the constraints $\alpha = \text{const}, \beta = \text{const}$ are imposed on the suspension. Using this notation, the characteristic equation can be represented in the form

$$p^2(p^2 + \nu_0^2) - \lambda^2 \frac{J_0}{I_0}(p^2 - \nu^2) = 0.$$

This characteristic equation has two pairs of purely imaginary roots; consequently, the solutions of our linear system will correspond to harmonic oscillations with two different natural frequencies. Figure 1 shows the dependence of the natural frequencies of the system on the magnitude λ.

For small values of λ the natural frequencies p_1 and p_2 are close to ν_0 and λ, respectively. It is of interest to study the behavior of the solutions of the linear equations when

as λ tends to zero and compare them with the solutions of the equations describing small oscillations of an ideal gyroscope.

Let us integrate the linear equations under the initial conditions

$$\xi = 0, \quad \dot{\xi} = \Omega, \quad \eta = 0, \quad \dot{\eta} = 0, \quad \zeta = 0, \quad \dot{\zeta} = 0.$$

Since the velocity u is small, the initial value $\dot{\xi}$ may be approximately taken equal to the value of the initial angular velocity of the outer ring. In the solutions obtained we shall decrease λ; then the amplitudes of the oscillations with frequency p_1 tend to the values of the amplitudes of the oscillations of an ideal gyroscope, while the amplitudes of the oscillations with frequency p_2 tend, for ξ, to zero, for η, to a finite limit, and for ζ the amplitude increases as $1/\lambda$. Thus, if we assume that $p_2 \simeq \nu_0,\ p_1 \simeq \lambda$, then the following approximate expressions may be written:

$$\xi = \frac{\Omega}{\nu_0} \sin \nu_0 t, \qquad \eta = \frac{H \cos \beta_0 \Omega}{(A + B_1)\nu_0^2} \left[\cos \lambda t - \cos \nu_0 t + \frac{J_0}{I_0}(1 - \cos \lambda t) \right],$$

$$\zeta = -\Omega \sin \beta_0 \left(\frac{1}{\nu_0} \sin \nu_0 t - \frac{1}{\lambda} \sin \lambda t \right).$$

The results obtained show that the solutions of the equations describing small oscillations of a synchronous gyroscope tend to the solutions of the corresponding equations for an ideal gyroscope as $\lambda \to 0$ nonuniformly in t, and that the amplitude of the oscillations of the angle θ reaches a large value for small λ. If, however, $f(\theta) = k \sin\theta$, then under the influence of nutational oscillations the gyroscope may sometimes fall out of synchronism.

We shall compute the drift velocity of a synchronous gyroscope during nutational oscillations, following the idea of S. S. Tikhmenev ([3]). The projections of the moment of the reaction forces in the supports of the outer-ring axis of the gimbal suspension onto the axes of the coordinate system $OXYZ$, computed with accuracy up to small quantities of third order, have the following constant components:

$$P'_Y = \frac{1}{2}\sum_{i=1,2}\left(A_2 + C_1 + \frac{\lambda^2}{\lambda^2 - p_i^2}C\right)\operatorname{tg}\beta_0\, p_i^2\overline{\xi_i^2}, \qquad P'_Z = 0,$$

where $\overline{\xi_1^2}$ and $\overline{\xi_2^2}$ are the amplitudes of the oscillations of the outer ring. According to the precessional theory of the gyroscope, the constant component of the moment of the reaction forces P'_Y must cause drift of the synchronous gyroscope with angular velocity

$$u = -\frac{P'_Y}{H\cos\beta_0} = -\frac{\sin\beta_0}{2H\cos^2\beta_0}\sum_{i=1,2}\left(A_2 + C_1 + \frac{\lambda^2}{\lambda^2 - p_i^2}C\right)p_i^2\overline{\xi_i^2}.$$

Because of the presence of friction forces, the nutational oscillations leading to gyroscope drift die out; therefore it is of interest to study forced oscillations caused, for example, by dynamic unbalance.

3. Forced oscillations under the action of dynamic unbalance. If the figure axis of the gyroscope does not coincide with the polar axis of inertia, then the gyroscope will execute small oscillations relative to the stationary rotation, having a forced character. The equations of motion of a dynamically unbalanced gyroscope may be formed by Lagrange's method. When the angle ε between the polar axis of inertia and the figure axis is zero, the equations of motion admit a particular solution corresponding to the stationary rotation of the gyroscope relative to a fixed axis. If the angle ε is small, then the terms containing ε will be regarded as small perturbations. We represent the perturbed motion in the form given in Sec. 2. Substituting the corresponding expressions for α, β, and θ into the equations of motion and retaining only the linear terms, we obtain the following system of linear differential equations:

$$J_0\ddot{\xi} + H\cos\beta_0\dot{\eta} + C\sin\beta_0\ddot{\zeta} = (C - A)\varepsilon\omega^2\cos\beta_0\cos(\omega t + \varphi),$$

$$I_1\ddot{\eta} - H\cos\beta_0\dot{\xi} = (C-A)\varepsilon\omega^2\sin(\omega t+\varphi),$$

$$C(\ddot{\zeta} + \ddot{\xi}\sin\beta_0) + k\zeta = 0,$$

where $\varphi = \gamma_0 - \theta_0$. For this system we find only the particular solution corresponding to forced oscillations:

$$\xi = -\left(1 - \frac{\lambda^2}{\omega^2}\right) Q\cos(\omega t+\varphi),$$

$$\eta = -\left[\left(1 - \frac{\lambda^2}{\omega^2}\right) R + S\right]\sin(\omega t+\varphi),$$

$$\zeta = Q\sin\beta_0\cos(\omega t+\varphi);$$

here the following notation has been introduced:

$$Q = \frac{(I_1 + C\cos\beta_0)\varepsilon(C-A)}{(I_0 I_1 - C^2\cos^2\beta_0) - (\lambda^2/\omega^2)(J_0 I_1 - C^2\cos^2\beta_0)},$$

$$R = \frac{(J_0 + C\cos^2\beta_0)\varepsilon(C-A)}{(I_0 I_1 - C^2\cos^2\beta_0) - (\lambda^2/\omega^2)(J_0 I_1 - C^2\cos^2\beta_0)},$$

$$S = \frac{C\sin\beta_0\varepsilon(C-A)}{(I_0 I_1 - C^2\cos^2\beta_0) - (\lambda^2/\omega^2)(J_0 I_1 - C^2\cos^2\beta_0)}.$$

Two interesting conclusions follow from the solutions obtained:

I. When the gyroscope rotates with angular velocity ω equal to λ, dynamic absorption of the oscillations of the outer gimbal ring occurs, i.e. $\xi \equiv 0$.

Fig. 2.

$$\operatorname{tg}\angle dO\omega = \frac{(A_2 + C_1 + C)\varepsilon^2(C-A)^2(I_1+C)^2}{2C\cos^2\beta_0\,(J_0 I_1 - C^2\cos^2\beta_0)^2}\sin\beta_0,$$

$$\operatorname{tg}\angle eO\omega = \frac{(A_2 + C_1)\varepsilon^2(C-A)^2(I_1+C)^2}{2C\cos^2\beta_0\,(J_0 I_1 - C^2\cos^2\beta_0)^2}\sin\beta_0$$

II. If $I_0 I_1 > C^2\cos^2\beta_0$, then when the gyroscope rotates with an angular velocity ω_1, slightly exceeding λ, resonance arises in the system; the critical velocity is determined by the formula

$$\omega_1 = \lambda \sqrt{\frac{J_0 I_1 - C^2 \cos^2 \beta_0}{I_0 I_1 - C^2 \cos^2 \beta_0}}.$$

We determine the drift rate of the gyroscope under forced oscillations by the method used in Sec. 2. The constant components of the projections of the moment of the reaction forces of the supports of the outer ring on the axes $OXYZ$ are:

$$P'_Y = \frac{1}{2}\left(A_2 + C_1 + \frac{\lambda^2}{\lambda^2 - \omega^2} C\right) \operatorname{tg} \beta_0 (\lambda^2 - \omega^2)^2 Q^2, \qquad P'_Z = 0.$$

Consequently, the angular velocity of precession of the gyroscope u can be represented in the form

$$u = -\omega \left(A_2 + C_1 + \frac{\lambda^2}{\lambda^2 - \omega^2} C\right) \frac{\left(1 - \frac{\lambda^2}{\omega^2}\right)^2 Q^2 \sin \beta_0}{2C \cos^2 \beta_0}.$$

The approximate form of the graph of the dependence of the absolute value of the angular velocity of precession on the magnitude of the rate of proper rotation is shown in Fig. 2. The line Od in this figure corresponds to the formula obtained in the work of D. M. Klimov ([4]).

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Note: Figure translations are in progress. See original paper for figures.