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Fig. 1 diagram

Figure 1: Fig. 1 diagram

Abstract

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A MATHEMATICAL MODEL OF SEPARATED FLOWS OF AN INCOMPRESSIBLE FLUID

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Let us attempt to model separated flows by means of a scheme of a certain "mixed" motion of an ideal fluid, which outside the separation zone is potential, while inside it has a constant vorticity ω, generally speaking unknown. To determine ω, additional conditions must be invoked. If, for example, a continuous flow with a finite velocity everywhere is impossible, then it is natural to impose a requirement analogous to Chaplygin' s postulate on the finiteness of the velocity at each critical point, or, what is the same thing, to require that the line separating the vortex region from the potential one pass through the points mentioned. In the present paper we shall regard the parameter ω as given.

§ 1. **Mathematical formulation of the problem.** Consider a flow in a plane bounded domain B with a piecewise smooth contour Γ. Introducing in the usual way the stream function Ψ, we impose on Γ the boundary condition: $\Psi|_\Gamma = \varphi(s)$. The function $\varphi(s)$ will be regarded as continuous, nonnegative, and different from zero only on a part of the contour ABC (Fig. 1). The problem consists in determining the line of separation γ of the domains of definition of two functions Ψ_1 and Ψ_2, satisfying the equations

$$\Delta\Psi_1 = 0; \qquad \Delta\Psi_2 = \omega, \tag{1,1}$$

Fig. 1

the boundary conditions

$$\Psi_1\big|_{EACF} = \varphi(s), \qquad \Psi_2\big|_{EKF} = 0$$

and the conditions on γ

$$\Psi_1 = \Psi_2 = 0; \qquad \frac{\partial \Psi_1}{\partial n} = \frac{\partial \Psi_2}{\partial n}.$$

Suppose that γ has been found. Then it is easy to see that under the conditions considered, $\Psi_1 \geqslant 0$, while $\Psi_2 \leqslant 0$. This property makes it possible to simplify the formulation of the problem. Namely, it is required to find a continuously differentiable function Ψ satisfying the equation

$$\Delta\Psi = f(\Psi) = \begin{cases} \omega, & \text{when } \Psi < 0, \\ 0, & \text{when } \Psi > 0 \end{cases} \tag{1,2}$$

and the boundary condition

$$\Psi\big|_{\Gamma} = \varphi(s). \tag{1,3}$$

Inverting the Laplace operator (^{1}), we obtain an integral equation equivalent to the problem (1,2)–(1,3):

$$\Psi(z) = \Psi_0(z) - \frac{\omega}{2\pi} \int_{B^-} \ln \frac{1}{|w(z,\zeta)|}\, d\xi\, d\eta. \tag{1,4}$$

Here $\Psi_0(z)$ is harmonic and satisfies condition (1,3); B^- is the region of negativity of the function Ψ; $w(z,\zeta)$ is an analytic function mapping the domain B onto the unit disk, whose center is the image of the point $\zeta \in B$; $\zeta = \xi + i\eta$; $z = x + iy$.

The formulation given above requires clarification, since problem (1.2)–(1.3) has the obvious solution $\Psi_0(z)$, which we shall call trivial. Since such a solution is unacceptable, the problem consists in finding conditions under which equation (1.4) has solutions different from the trivial one.

§ 2. **One-dimensional example.** In the one-dimensional case, problem (1.2) –(1.3) can be put in the form

$$\partial^2\Psi/\partial x^2 = f(\Psi); \qquad \Psi(0) = 0; \qquad \Psi(1) = 1. \tag{2.1}$$

Let us specify the function $f(\Psi)$ by

$$f(x) = \begin{cases} \omega, & x < \xi, \\ 0, & x > \xi; \end{cases}$$

then the solution of problem (2.1) is obtained in the form

$$\Psi = \begin{cases} x - \omega\xi\left(1 - \frac{1}{2}\xi\right)x + \frac{1}{2}\omega x^2, & x \le \xi, \\ x - \frac{1}{2}\omega\xi^2(1 - x), & x > \xi. \end{cases} \tag{2.2}$$

To determine the parameter ξ, impose the condition $\Psi(\xi) = 0$. This leads to the equation

$$\xi - \frac{1}{2}\omega\xi^2(1 - \xi) = 0, \tag{2.3}$$

which has three roots:

$$\xi_0 = 0; \qquad \xi_1 = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 8/\omega}; \qquad \xi_2 = \frac{1}{2} + \frac{1}{2}\sqrt{1 - 8/\omega}. \tag{2.4}$$

It follows from (2.4) that for $\omega < 8$ the problem has only the trivial solution $(\xi = 0, \psi = x)$; for $\omega > 8$ two more solutions appear, which at $\omega = 8$ merge: $\xi_1 = \xi_2 = \frac{1}{2}$. It is interesting to note that even in this case the nontrivial solutions are not close to the trivial one. As ω increases, the points ξ_1 and ξ_2 move apart in opposite directions; as $\omega \to \infty$, $\xi_1 \to 0$, $\xi_2 \to 1$. We note that, with respect to ω, equation (2.3) is uniquely solvable.

§ 3. **Existence theorem.** We shall prove that, for ω exceeding a certain value, equation (1.4) has at least one nontrivial solution. Denote

$$\Phi(\Omega) = \frac{1}{2\pi}\int_{\Omega} \ln\frac{1}{|w|}\,d\xi\,d\eta.$$

Draw a smooth arc γ_0, not intersecting ABC, which together with the part MN of the contour Γ bounds a domain B_0 (Fig. 1), and consider the function

$$\Psi_1 = \Psi_0 - \omega\Phi(B_0). \tag{3.1}$$

It is not difficult to verify that the functions Ψ_0 and Φ have the following properties: a) for $z \in B$, $\Psi_0 > 0$, $\Phi(B_0) > 0$; on the arc MN, $\Psi_0 = \Phi(B_0) = 0$; b) the function $\Phi(B_0)$, being an integral of Newtonian-potential type, is continuously differentiable inside the domain B, and outside B_0 it is harmonic (2); c) at all interior points of B, as well as at regular points of Γ, $|\nabla\Psi_0|$ and $|\nabla\Phi(B_0)|$ are bounded and different from zero; in a neighborhood of a corner point z_0 of the contour Γ, characterized by an angle α, $|\nabla\Psi_0|$ and $|\nabla\Phi(B_0)|$ have the same asymptotics:

$$|\nabla\Psi_0|,\ |\nabla\Phi(B_0)| \sim C|z - z_0|^{(\pi-\alpha)/\alpha}.$$

On the basis of the properties listed, one can assert that the ratio $\Psi_0/\Phi(B_0)$ remains bounded everywhere in the closed domain $\overline{B}_0$.

Choose the number ω so large that the inequality

$$\omega > \sup_{z\in\overline{B}0} \frac{\Psi_0}{\Phi(B_0)}$$

is satisfied. Then, according to (3.1), everywhere in $\overline{B}_0$, $\Psi_1 \leq 0$, where equality is attained only at points belonging to the contour Γ. Since near ABC, $\Psi_1 > 0$, there exists a line γ_1 (a level line of a harmonic function) on which $\Psi_1 = 0$. It is easy to see that γ_1 cannot have branch points inside B.

Define the function

$$\Psi_2 = \Psi_0 - \omega\Phi(B_1); \tag{3.2}$$

where ω is already fixed, and B_1 is the domain of negativity of the function Ψ_1. Since γ_1 lies entirely outside B_0, $B_1 \supset B_0$; therefore one may set $B_1 = B_0 + \Delta_1$, where $\operatorname{mes}\Delta_1 > 0$. From (3.2) we obtain $\Psi_2 = \Psi_1 - \omega\Phi(\Delta_1)$,

whence it follows that $\Psi_2 \leqslant \Psi_1$, with equality attained only at points of the contour Γ. Consequently, on the line γ_1, $\Psi_2 < 0$, and the line γ_2, on which $\Psi_2 = 0$, lies entirely outside the domain B_1, so that $B_2 \supset B_1$. Further, by induction it is not difficult to prove that the functions

$$\Psi_n = \Psi_0 - \omega\Phi(B_{n-1}) \tag{3.3}$$

and the domains B_n form monotone sequences:

$$\Psi_1 \geqslant \Psi_2 \geqslant \cdots \geqslant \Psi_n \geqslant \cdots; \quad B_0 \subset B_1 \subset \cdots \subset B_n \subset \cdots \tag{3.4}$$

On the other hand, $B_n \subset B$, and $\Psi_n \geqslant \Psi_0 - \omega\Phi(B)$, as follows immediately from (3.3). Hence we conclude that there exist

$$\lim_{n\to\infty} \Psi_n = \Psi_*; \qquad \lim_{n\to\infty} B_n^- = B_*. \tag{3.5}$$

Let us prove that the limiting function is continuously differentiable. To this end, rewrite (3.3) in the form $\Psi_n = \Psi_0 - \omega\Phi(B_*) + q_n$; $q_n = \omega\Phi(\Delta)$, $\Delta = B_* - B_{n-1}$. According to (3.5), as $n \to \infty$, $\operatorname{mes}\Delta \to 0$, and therefore $q_n \to 0^{(2)}$. Thus, in the limit,

$$\Psi_* = \Psi_0 - \omega\Phi(B_*). \tag{3.6}$$

It follows at once from this that Ψ_* is continuously differentiable. On the basis of Dini' s theorem, the continuity of Ψ_* implies the uniform convergence of the monotone sequence $\{\Psi_n\}$ and hence the equicontinuity of the functions Ψ_n. These properties make it possible to prove that $\Psi_* \leqslant 0$ if $z \in B_*$, and that on the boundary γ of the domain B_*, $\Psi_* = 0$*. Thus, B_* coincides with the domain of negativity of the function Ψ_*, which permits us to put $\Psi_* = \Psi$. A nontrivial solution of equation (1.4) for sufficiently large values of ω has been constructed.

On the other hand, for sufficiently small ω, equation (1.4) can have only the trivial solution, as follows from the principle of contraction mappings. Consequently, there exists a value of the parameter ω_0 at which the solution branches for the first time. Owing to the a priori boundedness of all solutions of (1.4), for any finite ω the number of solutions for each ω is odd $^{(3)}$. This means that for any $\omega > \omega_0$ there exists a second nontrivial solution. The question of whether the number of solutions is exhausted by the three found remains open.

§ 4. **Some properties of the solutions.** Let the value ω_1 correspond to a nontrivial solution Ψ_1. We shall show that among the solutions corresponding to the parameter $\omega_2 > \omega_1$ there is a solution $\Psi_2 < \Psi_1$, and consequently $B_2^- \supset B_1^-$. Indeed, $\Psi_1 = \Psi_0 - \omega_1 \Phi(B_1^-)$. Consider the function $\Psi_2^{(1)} = \Psi_0 - \omega_2 \Phi(B_1^-)$. It is obvious that $\Psi_2^{(1)} - \Psi_1 < 0$, whence it follows that $B_2^{(1)} \supset B_1^-$. Arguing exactly as in the proof of the existence theorem, it is not difficult to establish that the sequence $\{\Psi_2^{(n)}\}$ converges monotonically to a function Ψ_2 satisfying equation (1.4) with parameter $\omega = \omega_2$, and, by the method of construction, $\Psi_2 < \Psi_1$ and $B_2^- \supset B_1^-$. It is thereby proved that, as ω increases, the domain of negativity of one of the solutions expands monotonically, tending to cover the entire domain B. By investigating the boundary derivatives of the function Ψ, one can prove that, as ω increases, the endpoints of the γ-line move continuously along the contour Γ in the direction of the points A and C.

* In fact, take some point z_0 on γ. In an arbitrarily small neighborhood of it, points belonging to B_n, and therefore also to γ_n, must be found, by (3.5), in a set of nonzero measure. For any point z_n on γ_n, $\Psi_n(z_n) = 0$. Given an arbitrarily small $\varepsilon > 0$, choose a number N such that for $n > N$ one has $|\Psi_n(z_0) - \Psi_*(z_0)|$, $|\Psi_n(z_n) - \Psi_n(z_0)| < \varepsilon/2$. This is possible owing to the uniform convergence of the sequence $\{\Psi_n\}$ and the equicontinuity of the functions Ψ_n. Since by assumption $\Psi_n(z_n) = 0$, it follows that $|\Psi_n(z_0)| < \frac{1}{2}\varepsilon$, and consequently $|\Psi_*(z_0)| < \varepsilon$; since $|\Psi_*(z_0)|$ does not depend on ε, it follows that $|\Psi_*(z_0)| = 0$.

Consider the functional for which the Euler equation coincides with (1.2):

$$I(\Psi) = \int_B \left[(\nabla\Psi)^2 + 2\omega R(\Psi)\right]\, dx\, dy; \qquad R(\Psi) = \begin{cases} \Psi, & \Psi < 0, \\ 0, & \Psi \geq 0. \end{cases} \tag{4,1}$$

As admissible functions we shall take solutions of the linear problem

Fig. 2

Figure 2: Fig. 2

$$\Delta\Psi = f(z) = \begin{cases} \omega, & z \in B^-; \\ 0, & z \in \overline{B^-}; \end{cases} \qquad \Psi\big|_{\Gamma} = \varphi(s) \tag{4,2}$$

for all possible domains B^-, and pose the problem of finding stationary values of the functional (4.1). In the one-dimensional case the solution of problem (4.2) has already been obtained in the form (2.2). Substitution of (2.2) into (4.1) gives $I(\Psi) = 1 + \omega\xi^2 - \frac{1}{3}\omega^2\xi^2 + \frac{1}{4}\omega^2\xi^4$. Hence it is seen that the condition $dI/d\xi = 0$ leads to an equation coinciding with (2.3). It is not difficult to verify that $I(\xi)$ has a maximum at the point ξ_1, and minima at the points ξ_0 and ξ_2, with $I(\xi_2) < I(\xi_0)$.

Fig. 2

Thus, the following property has been found: each of the extremals (2.2) for $\xi > 1/2$ gives a minimum of the functional I, but the smallest of all the minimal values is attained at $\xi = \xi_2$, when the discontinuity point of the function $f(x)$ coincides with the zero of the solution $\Psi(x)$, which in this case satisfies the nonlinear equation (1.2). It can be proved that this property is also inherent in the planar problem.

§ 5. **Example.** As an example, let us consider the flow past a "cavity" of square cross-section by a stream having unit velocity at infinity. The results of computations on the BESM-2 electronic computer by the iteration method are presented in Fig. 2. In fact, the computations were carried out for the finite domain $EMADCBNFE$ ($EM = MA = AD = DC = 1$) under the following boundary conditions: on EF, $\Psi = 1$; on $MADCBN$, $\Psi = 0$; on ME and NF, $\Psi = y$. The number ω was chosen so that the γ-line passed through the points A and B. It was found that $\omega = 4.60$, $\Psi_{\min} = -0.332$.

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Note: Figure translations are in progress. See original paper for figures.