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Abstract

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ON SPLITTING IN THE KÜNNETH FORMULA

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It is well known that the Künneth formula

$$0 \to H(K) \otimes H(L) \to H(K \otimes L) \to \mathrm{Tor}(H(K),\, H(L)) \to 0$$

splits if the complexes (i.e., graded differential groups, or even only differential groups with involution) K and L are free (as abelian groups): the splitting

$$H(K \otimes L) \approx H(K) \otimes H(L) + \mathrm{Tor}(H(K),\, H(L))$$

is achieved here by singling out the groups of cycles as direct summands. In our paper ([1]) we proved a considerably stronger assertion: that the Künneth formula splits for arbitrary complexes K and L having no elements of finite order*; here the splitting is achieved in a far more complicated way, by considering the homology spectra of these complexes, since, as we showed in ([2]) (or ([3,4])), for complexes without elements of finite order the homology spectra are determined, up to isomorphism (not a natural one), by their homology groups. However, as is known, the Künneth formula in the form of an exact sequence is also valid in the case when only one of the complexes K or L has no elements of finite order, and so the question arises whether it will split in this case. We shall now give an example showing that such a splitting need not occur (this is apparently the simplest of such examples).

Consider two complexes

$$K:\ 0 \to Z \xrightarrow{d} Z \to 0 \quad \text{and} \quad L:\ 0 \to Z_8 \xrightarrow{d} Z_8 \to 0,$$

where in the first complex $d1 = 2$, and in the second $d1_8 = 2_8$ (by 1_8 and 2_8 we have denoted the residue classes modulo 8 containing, respectively, the numbers 1 and 2). Their homology groups are, obviously, as follows (when they are graded from left to right by the numbers $-1, 0, 1, 2$):

$$H^0(K) = 0, \qquad H^1(K) = Z_2; \qquad H^0(L) = Z_2, \qquad H^1(L) = Z_2.$$

The tensor product of these complexes is the complex

$$K \otimes L: \quad 0 \to Z \otimes Z_8 \to Z \otimes Z_8 + Z \otimes Z_8 \to Z \otimes Z_8 \to 0,$$

i.e.

$$K \otimes L: \quad 0 \to Z_8(e) \xrightarrow{d} Z_8(a) + Z_8(b) \xrightarrow{d} Z_8(c) \to 0,$$

where in parentheses (as also everywhere below) we indicate generators of the cyclic groups Z_8, and $de = 2a + 2b,\ da = 2c,\ db = 2c$. Therefore, if $x = \lambda a + \mu b$, then $dx = 2(\lambda + \mu)c$, so that $dx = 0$ (i.e., the fact that x is a cycle) means that $\lambda + \mu \equiv 0 \pmod 4$. Hence the subgroup of cycles in the group

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* Taking this opportunity, I note an inaccuracy in one place of the reasoning in that paper, which, however, has no significance for the proof: at the beginning of p. 1190 one should speak not of an isomorphic embedding, but of a homomorphic mapping.

$(K \otimes L)^1 = Z_8(a) + Z_8(b)$ is a direct sum, $\mathfrak{Z} = \mathfrak{Z}^1(K \otimes L) = Z_8(3a + b) + Z_2(4a)$, and the subgroup of boundaries is $\mathfrak{B} = \mathfrak{B}^1(K \otimes L) = Z_4(2a + 2b)$, so that $H^1(K \otimes L) = \mathfrak{Z}/\mathfrak{B} \simeq Z_4$ (the generating element of this quotient group is then the homology class containing the element $3a + b$ of the group of cycles, whereas the element $4a$ lies in twice this class). Thus, in the example under consideration the Künneth formula can be written only in the form of the exact sequence $0 \to Z_2 \otimes Z_2 \to Z_4 \to \mathrm{Tor}(Z_2, Z_2) \to 0$ (for dimension 1), but not in the form of a direct sum.

Nevertheless, the Künneth formula for the tensor product of two complexes, of which only one has elements of finite order, can split in some cases. For example, as we showed in paper ([2]) (or ([3],[4])), it always splits in the case where the tensor factor having elements of finite order has trivial differential d, i.e. can be regarded as an ordinary, and not a differential, abelian group (in this case the Künneth formula turns into the formula of universal coefficients). The reason why the arguments given there are not suitable in the general case indicated here (as regards the method of paper ([1]), its inapplicability in the case where one of the complexes being multiplied has elements of finite order is evident) is that the analogue of the isomorphism Φ, defined there, from the right-hand side of formula (4) of paper ([2]) to its left-hand side ceases to be a single-valued mapping.

Indeed, let L be a complex without elements of finite order, and let G be an arbitrary complex, $h \in H^q(L, Z_m)$, $u \in H^r(G)$, $mu = 0$, and let l be a cycle modulo m from h, i.e. $dl = m\bar{l}$, so that $\bar{l} = \frac{1}{m} dl$ (for division by m in the complex L is possible only in a unique way, because of the absence of elements of finite order), $\bar{g} \in u$, $m\bar{g} \sim 0$, i.e. $m\bar{g} = dg$. If now to the element $h \otimes u$ we assign the homology class of the complex $L \otimes G$ containing the cycle $z = \bar{l} \otimes g + (-1)^q l \otimes \bar{g}$ (this is a cycle, since $dz = d\bar{l} \otimes g + (-1)^{q+1} \bar{l} \otimes dg + (-1)^q dl \otimes \bar{g} + l \otimes d\bar{g} = 0 + (-1)^{q+1} \bar{l} \otimes m\bar{g} + (-1)^q ml \otimes \bar{g} + 0 = 0$), then the admissible changes of l do not lead to a change of this homology class, since if $l \sim l'$ mod m, i.e. $l' = l + d\tilde{l} + m\tilde{\tilde{l}}$, $\bar{l}' = \frac{1}{m} dl'$, then

$$(\bar{l}' \otimes g + (-1)^q l' \otimes \bar{g}) - (\bar{l} \otimes g + (-1)^q l \otimes \bar{g}) = (\bar{l}' - \bar{l}) \otimes g + (-1)^q (l' - l) \otimes \bar{g}$$

$$= \frac{1}{m} d(l' - l) \otimes g + (-1)^q (l' - l) \otimes \bar{g} = \frac{1}{m} d(d\tilde{l} + m\tilde{\tilde{l}}) \otimes g + (-1)^q (d\tilde{l} + m\tilde{\tilde{l}}) \otimes \bar{g}$$

$$= d\tilde{\tilde{l}} \otimes g + (-1)^q d\tilde{l} \otimes \bar{g} + (-1)^q \tilde{\tilde{l}} \otimes mg = d(\tilde{\tilde{l}} \otimes g + (-1)^q \tilde{l} \otimes \bar{g}) \sim 0.$$

In exactly the same way this class will not change if one changes $\bar{g}$, replacing it by a cycle $\bar{g}'$ homologous to it, i.e. such that $\bar{g}' = \bar{g} + d\tilde{g}$, and putting at the same time $g' = g + m\tilde{g}$ (since $m\bar{g}' = m\bar{g} + md\tilde{g} = d(g + m\tilde{g})$), because

$$(\bar{l} \otimes g' + (-1)^q l \otimes \bar{g}') - (\bar{l} \otimes g + (-1)^q l \otimes \bar{g}) = \bar{l} \otimes (g' - g) + (-1)^q l \otimes (\bar{g}' - \bar{g})$$

$$= \bar{l} \otimes (g' - g) + (-1)^q l \otimes d\tilde{g} = \bar{l} \otimes (g' - g) + d(l \otimes \tilde{g}) - dl \otimes \tilde{g} = \bar{l} \otimes (g' - g - m\tilde{g}) + d(l \otimes \tilde{g}) = d(l \otimes \tilde{g}) \sim 0.$$

However, if (leaving $\bar{g}$ fixed) one changes g, replacing it by another cycle g', for which also $m\bar{g} = dg'$, then the homology class under consideration will already change, since now the previous transformation leads to the cycle $\bar{l} \otimes (g' - g - m\tilde{g}) = \bar{l} \otimes (g' - g)$ (as $\tilde{g} = 0$), which is no longer homologous to zero. True, this is the same class modulo the classes generated by cycles that are tensor products of cycles of the complexes L and G (since $d\bar{l} = d\left(\frac{1}{m} dl\right) = 0$, $d(g' - g) = m\bar{g} - m\bar{g} = 0$), but such a consideration obviously leads only to a non-splitting Künneth formula. Hence it is clear that the requirement of triviality of the differen-

of the differential d in the complex G (in this case one may, obviously, without loss of generality regard the grading as trivial), can only be weakened (replacing it, for example, by the requirement of homological triviality, i.e. vanishing of the homology groups in dimensions different from the dimension r under consideration; in this case $g' - g \sim 0$, i.e. $g' - g = d\tilde{g}$, $\tilde{l} \otimes (g' - g) = \pm d(\tilde{l} \otimes \tilde{g}) \sim 0$), and it is clear why it cannot be discarded altogether.

Let us note that an analogous state of affairs holds in the case of the tensor product of several complexes. Namely, using the method of the paper ([1]) and relying here on the results of our paper ([5]), we succeeded in obtaining the following generalization of MacLane' s results ([6]). If $K_1, \dots, K_n$ are complexes without elements of finite order, then there exists a natural filtration

$$0 = S_0 \subset S_1 \subset S_2 \subset \cdots \subset S_n = H(K_1 \otimes \cdots \otimes K_n)$$

and one can define covariant additive functors $\mathrm{Mult}_i(A_1, \dots, A_n)$ $(i = 1, \dots, n)$, which assign to each system of n graded abelian groups $A_1, \dots, A_n$ a new graded abelian group, so that there is a natural isomorphism

$$\mathrm{Mult}_i(H(K_1), \dots, H(K_n)) \approx S_i / S_{i-1}.$$

At the same time there is also an isomorphism (no longer natural)

$$S_1 + S_2/S_1 + \cdots + S_n/S_{n-1} \approx H(K_1 \otimes \cdots \otimes K_n),$$

which, obviously, may be regarded as a splitting of the preceding assertion.

The groups $\mathrm{Mult}_i(A_1, \dots, A_n)$ are defined by the formula

$$\mathrm{Mult}_i(A_1, \dots, A_n) = \left[\sum \mathrm{Tor}(A_{s_1}, \dots, A_{s_i}) \otimes A_{r_1} \otimes \cdots \otimes A_{r_{n-i}}\right] / E_i,$$

in which the direct summation extends over all mutually distinct values of the numbers $s_1, \dots, s_i, r_1, \dots, r_{n-i}$ for which $s_1 < \cdots < s_i$, $r_1 < \cdots < r_{n-i}$, while the subgroup E_i is the zero group for $i = 1$ and $i = n$, and for the remaining values of i is generated by all elements of the form

$$\partial\tau_h(a_{s_1}, \dots, a_{s_{i+1}}) \otimes a_{r_1} \otimes \cdots \otimes a_{r_{n-i-1}},$$

where $\partial\tau_h$ is defined by the relation ($q'_\nu = \dim a_\nu - 1$)

$$\partial\tau_h(a_1, \dots, a_k) = \sum_{\nu=1}^{k} (-1)^{q'_\nu(q'_{\nu+1} + \cdots + q'_k + 1)} \tau_h(a_1, \dots, a_{\nu-1}, a_{\nu+1}, \dots, a_k) \otimes a_\nu$$

(here one should keep in mind the possibility of permuting the tensor factors with the usual change of sign thereby defined), and $\tau_h(a_1, \dots, a_m)$ are the generators of the group $\mathrm{Tor}(A_1, \dots, A_m)$, defined as in MacLane' s paper ([6]). As for the group S_i, it is that subgroup of the group $H(K_1 \otimes \cdots \otimes K_n)$ which is generated by homology classes containing cycles of the form

$$z_{s_1 \dots s_i} \otimes z_{r_1} \otimes \cdots \otimes z_{r_{n-i}},$$

where $z_{s_1 \dots s_i}$ is a cycle from the tensor product $K_{s_1} \otimes \cdots \otimes K_{s_i}$ ($s_1 < \cdots < s_i$) of some i of the given complexes, and $z_{r_1}, \dots, z_{r_{n-i}}$ are cycles from the remaining complexes $K_{r_1}, \dots, K_{r_{n-i}}$ ($r_1 < \cdots < r_{n-i}$).

Combining the methods of our papers ([1,2]), one can show that this result remains valid if one of the complexes being multiplied is replaced by an abelian group, which may already have elements of finite order (but is not a differential group). If, however, we have the tensor product of n complexes, of which all but one have no elements of finite order, then the splitting assertion in the preceding statement will, generally speaking, no longer be valid (it suffices to put

$K_1 = \cdots = K_{n-2} = Z$, and for K_{n-1} and K_n take, respectively, the complexes K and L constructed by us at the beginning of the present paper), whereas the remaining part of the assertion retains its force. Thus, the situation here is the same as in the case of the ordinary Künneth formula.

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Note: Figure translations are in progress. See original paper for figures.