Mathematics

D. L. Berman

1962

SovietRxiv

Abstract

Full Text

Mathematics

D. L. Berman

CLASSIFICATION OF POLYNOMIAL OPERATIONS

(Presented by Academician S. N. Bernstein on 2 February 1962)

1°. Let $\widetilde{C}$ denote the space of all continuous 2π-periodic functions $f(x)$ with norm

$$\|f(x)\| = \max_{0 \le x < 2\pi} |f(x)|;$$

let Π_n denote the set of all trigonometric polynomials of order $\le n$; let M_n denote the set of all linear operations from $\widetilde{C}$ to $\widetilde{C}$ that map $\widetilde{C}$ into Π_n. Let $U_n \in M_n$. Introduce the operator

$$\widetilde{U}_n(f, x) = \frac{1}{2\pi} \int_0^{2\pi} U_n(f_t, x - t)\, dt,$$

where $f_t(x) = f(x + t)$. It is easy to see that $\widetilde{U}_n$ also belongs to M_n.

2°. Let $U_n^{(i)} \in M_n$, $i = 1, 2$. We say that $U_n^{(1)}$ and $U_n^{(2)}$ belong to one and the same class if $\widetilde{U}_n^{(1)} = \widetilde{U}_n^{(2)}$. Thus the set M_n is partitioned into classes. The set of all classes will be denoted by Σ_n. It is noteworthy that, under this classification, the partial sum of the Fourier series $S_n(f, x)$ and the Lagrange interpolation polynomial $L_n(f, x)$, constructed for an arbitrary system of nodes

$$0 \le x_0 < x_1 < \cdots < x_{2n} < 2\pi,$$

belong to one and the same class, since $\widetilde{L}_n = \widetilde{S}_n$ ([1]). It is easy to see that the classes are pairwise disjoint.

Theorem 1. *Between the set of all classes Σ_n and the set of all trigonometric polynomials of order not exceeding n, Π_n, there is a one-to-one correspondence. This mapping is an isomorphism if the composition of elements in Σ_n and Π_n is defined in the corresponding manner.*

We give the main points of the proof. Let $T_n \in \Pi_n$. Introduce the convolution σ_n with kernel T_n:

$$\sigma_n = \sigma_n(f, T_n, x) = \int_0^{2\pi} f(x + t) T_n(t)\, dt. \quad (1)$$

Obviously, $\sigma_n \in M_n$, and therefore there exists a unique class $K_n \in \Sigma_n$ containing σ_n. To the polynomial T_n we assign the class K_n. We denote this mapping of Π_n into Σ_n by φ. Now map Σ_n into Π_n. Let K'_n be any element of Σ_n. To the class K'_n we assign the polynomial

$$T'_n(t) = \widetilde{U}_n(D_n, -t),$$

where U_n is an arbitrary operator from K'_n. Here D_n denotes the Dirichlet kernel of order n. We denote the indicated mapping of Σ_n into Π_n by ψ. We now show that φ is a one-to-one mapping. Let $\varphi(T_n) = K_n$. According to the definition of φ, this means that the convolution $\sigma_n \in K_n$. One can prove that, for any $U_n \in K_n$, the equalities

$$\widetilde{U}_n = \tilde{\sigma}_n = \sigma_n$$

hold. Note also that

$$\tilde{\sigma}_n(D_n, -t) = \int_0^{2\pi} D_n(-t + t_1) T_n(t_1)\, dt_1 = T_n(t).$$

Therefore, in accordance with the definition of the mapping ψ, we have that

$\psi(K_n) = T_n$. Suppose now that $\psi(K_n^{(2)}) = T_n^{(2)}$, where $K_n^{(2)} \in \Sigma_n$ and $T_n^{(2)} \in \Pi_n$. According to the definition of ψ,

$$T_n^{(2)}(t) = \widetilde{U}_n(D_n, -t), \qquad U_n \in K_n^{(2)}.$$

Consider the convolution

$$\sigma_n^{(2)}(f, x) = \int_0^{2\pi} f(x + t) T_n^{(2)}(t)\, dt. \tag{1'}$$

Since $\sigma_n^{(2)} = \tilde{\sigma}_n^{(2)} = \widetilde{U}_n$, it follows that $\sigma_n^{(2)} \in K_n^{(2)}$. According to the definition of φ, this means that $\varphi(T_n^{(2)}) = K_n^{(2)}$. Consequently, φ is a one-to-one correspondence.

It is not difficult to verify that each class K_n contains only one convolution. Therefore the class containing the convolution σ is naturally denoted by K_σ. Obviously, if the composition of two elements T_1 and T_2 of Π_n is defined as their sum $T_1 + T_2$, and the composition of two classes K_{σ_1} and K_{σ_2} is defined as the class containing the convolution $\sigma_1 + \sigma_2$, then the mapping φ becomes an isomorphism. It is curious that the mapping φ is an isomorphism also in the case when the composition of two elements $T_i \in \Pi_n$, $i = 1, 2$, is defined as their convolution $T_1 * T_2$, while the composition of the classes K_{σ_1} and K_{σ_2} is defined as the class containing the convolution $\sigma_2\sigma_1$. Let us consider some class K_σ and pose the question of finding in K_σ an operator with the smallest norm. The answer to this question is given by Theorem 2.

Theorem 2. Among the various operations of the class K_σ, the convolution σ has the smallest norm, i.e.

$$\inf_{U_n \in K_\sigma} \|U_n\| = \|\sigma\|. \tag{2}$$

Proof. According to the property of the operator $\widetilde{U}_n$, $\|\widetilde{U}_n\| \leq \|U_n\|$ ([2]). It has already been noted that for any $U_n \in K_\sigma$, $\widetilde{U}_n = \tilde{\sigma} = \sigma$. Therefore (2) is valid. It is useful to compare this theorem with the results in ([4–7]).

3°. Theorem 2 suggests the validity of the following theorem:

Theorem 3. Let the sequence $\{U_{n_j}(f,x)\}_{j=1}^{\infty}$, where $U_{n_j} \in M_{n_j}$, $j = 1, 2, \ldots$, converge uniformly for every $f \in \widetilde{C}$,

$$U_{n_j}(f,x) \to f(x), \qquad n_j \to \infty. \tag{3}$$

Then for every $f \in \widetilde{C}$ the relation

$$\widetilde{U}_{n_j}(f,x) \to f(x), \qquad n_j \to \infty. \tag{3'}$$

holds uniformly.

In proving Theorem 3 we use the following lemma:

Lemma. Let a sequence of linear operations from $\widetilde{C}$ into $\widetilde{C}$, $\{U_n(f,x)\}_{n=1}^{\infty}$, satisfy, for every $f \in \widetilde{C}$, the uniform relation (3). Then for every $\varepsilon > 0$ there exists an N such that, for every $t \in [0, 2\pi]$, the inequality

$$\|U_n(f_t) - f_t\| < \varepsilon, \qquad n \geq N, \tag{4}$$

holds, where N does not depend on t.

The proof of Theorem 3 is carried out as follows. From the definition of $\widetilde{U}_n$ we obtain

$$\widetilde{U}_{n_j}(f,x) - f(x) = \frac{1}{2\pi}\int_0^{2\pi} [U_{n_j}(f_t) - f_t,\ x - t]\, dt, \tag{5}$$

where $[f, x]$ is the value of the function f at the point x. Hence we obtain

$$\|\widetilde{U}_{n_j}(f) - f\| \leq \frac{1}{2\pi}\int_0^{2\pi} \|U_{n_j}(f_t) - f_t\|\, dt. \tag{6}$$

The justification of the validity of the transition from (5) to (6) is given in ([2]). From inequalities (4) and (6) we obtain that

$$\|\widetilde{U}_{n_j}(f) - f\| < \varepsilon, \qquad n_j > N.$$

Since for every $U_n \in K_\sigma$ one has $\widetilde{U}_n = \sigma$, relation (3′) can be written in the form

$$\sigma_{n_j}(f, x) \to f(x), \qquad n_j \to \infty.$$

Thus, Theorem 3 can be formulated in the following equivalent form:

Theorem 4. Let the sequence of operations $\{U_{n_j}(f, x)\}_{j=1}^{\infty}$ converge uniformly for every $f \in \widetilde{C}$. Then one can construct a sequence of convolutions of the form (1′), $\{\sigma_{n_j}\}_{j=1}^{\infty}$, which also converges uniformly for every $f \in \widetilde{C}$.

4°. Let a certain fixed class of operators $K_n \in \Sigma_n$ be given, and let U_n be a certain fixed operator from K_n. Denote by $M_{n+m}(U_n)$ the set of all possible linear operators from $\widetilde{C}$ into $\widetilde{C}$ having the properties: 1) for every $f \in \widetilde{C}$, $U_{n+m}(f) \in \Pi_{n+m}$; 2) if $f \in \Pi_n$, then $U_{n+m}(f) = U_n(f)$ $(m \geq 0)$. A special case of such operations was considered in ([3]).

Theorem 5. Let $U_{n+m} \in M_{n+m}(U_n)$; then for every $f \in \widetilde{C}$ the equality holds

$$\widetilde{U}_{n+m}(f, x) = \int_0^{2\pi} f(x+t) \Big[\widetilde{U}_n(D_n, -t) +$$

$$+ \frac{1}{2\pi} \sum_{k=n+1}^{n+m} (\alpha_k \cos kt + \beta_k \sin kt)\Big] f(x+t)\, dt,$$

where

$$\alpha_k = s_k^{(2)} + c_k^{(1)},$$

$$\beta_k = a_k s_k^{(1)} - c_k^{(2)}.$$

Here $c_k^{(1)}$ and $c_k^{(2)}$, $s_k^{(1)}$ and $s_k^{(2)}$ are the Fourier coefficients of order k, respectively, for the functions $U_{n+m}(\cos kz, x)$, $U_{n+m}(\sin kz, x)$.

With the aid of Theorem 5 one proves

Theorem 6. For every class $K_n \in \Sigma_n$ the equality holds

$$\inf_{U_n\in K_n}\ \inf_{U_{n+m}\in M_{n+m}(U_n)} \|U_{n+m}\| =$$

$$= \inf_{\alpha_k,\beta_k} \int_0^{2\pi} \left| \widetilde{U}_n(D_n,t) + \sum_{k=n+1}^{n+m} (\alpha_k \cos kt + \beta_k \sin kt) \right| dt.$$

Leningrad Institute of Soviet Trade
named after F. Engels

Received
18 I 1962

CITED LITERATURE

[1] J. Marcinkiewicz, *Acta Litt. Sci. Szeged*, **8**, 127 (1937).
[2] D. L. Berman, *Izv. Vyssh. uchebn. zaved., Matematika*, **4** (17) (1960).
[3] D. L. Berman, *DAN*, **95**, No. 2 (1954).
[4] S. M. Lozinskii, *DAN*, **61**, No. 2 (1948).
[5] S. M. Lozinskii, *DAN*, **89**, No. 5 (1953).
[6] D. L. Berman, *DAN*, **85**, No. 1 (1952).
[7] D. L. Berman, *DAN*, **88**, No. 1 (1953).