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Notation.

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Abstract

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ON THE ANALYTIC ARITHMETIC OF QUADRATIC FORMS WITH AN ODD NUMBER OF VARIABLES IN CONNECTION WITH THE THEORY OF MODULAR FORMS

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Notation.
$f(x_1,\dots,x_m)=\sum\limits_{1\le i\le j\le m} a_{ij}x_{ix}j$ is a positive definite quadratic form with integral coefficients and an odd number of variables $m=2k+1$; D is the discriminant of f; q is the level of f; l is a natural number not divisible by a square different from 1; $r_f(n)$ is the number of integral representations of the number n by the form f; t_0 is the least natural number such that $r_f(lt_0^2)>0$; $q_1=4lt_0^2q$; $q_2=16l^3t_0^6q$; $\Gamma(N)$ (respectively $\Gamma(N,N_0)$ for $N\equiv 0 \pmod{N_0}$) is the group of integral unimodular matrices of order two $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ such that $\gamma\equiv 0 \pmod N$ (respectively $\gamma\equiv 0 \pmod N$; $\beta\equiv 0 \pmod{N_0}$); $\left(\frac{P}{Q}\right)$ is the Jacobi symbol;

$$g_l(t)=\sum_{\substack{d\mid t\\ (d,t_0)=1}} d^{k-1}\left(\frac{2lD}{d}\right) r_f\left(\frac{lt^2}{d^2}\right);$$

τ is a complex variable.

In this note the following is proved.

Theorem. The function

$$G_l(\tau)=\sum_{t=1}^{\infty} g_l(t)\exp\left(\frac{2\pi i\tau t}{t_0}\right),$$

regular in the upper half-plane $\operatorname{Im}\tau>0$, is represented there in the form

$$G_l(\tau)=C_l+F_l(\tau)+\widetilde{F}_l(\tau),$$

where C_l is a constant; $F_l(\tau)$, $\widetilde{F}_l(\tau)$ are functions regular for $\operatorname{Im}\tau>0$, and

$$F_l\left(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right)=(\gamma\tau+\delta)^{2k}F_l(\tau)\quad\text{for}\quad\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in\Gamma(q_2,t_0),$$

$$\widetilde{F}_l\left(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right) = \left(\frac{2lD}{\alpha}\right)(\gamma\tau+\delta)^k\widetilde{F}_l(\tau) \quad \text{for} \quad \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in \Gamma(q_1).$$

From this theorem, on the basis of the results of the work (^{1}), there follows the following

Corollary. Let f be a positive definite ternary form; then there exists a number $Q = Q_{l,f}$ such that for all t relatively prime to Q, an asymptotic formula holds for the number of integral representations of the number lt^2 by the ternary form f, or, what is the same, for the number of rational points on the ellipsoid $f(x_1, x_2, x_3) = l$ with denominator t:

$$r_f(lt^2) = C_{lf}\sum_{d|t} d \prod_{p|d}\left(1 - \left(\frac{2lD}{p}\right)\frac{1}{p}\right) + \alpha_{lf}(t), \tag{*}$$

where

$$|\alpha_{lf}(t)| < C'_{lf}\tau^2(t)\sqrt{t};$$

Here C_{lf}, C'_{lf} are certain constants; $\tau(t)$ is the number of divisors of the number t; p runs through the prime divisors d.

Formula (*) may be regarded as a generalization of Gauss' s formulas for the number of integral representations of an integer as a sum of three squares.

Proof of the theorem.

1°. Put

$$\varphi(x_2, \dots, x_m) = 4a_{11}\sum a_{ij}x_ix_j - (a_{12}x_2 + \dots + a_{1m}x_m)^2.$$

Denote by D_φ and q_φ the discriminant and the level of the form φ. It is easy to see that $D_\varphi = 2^{2k-1}a_{11}^{2k-1}D$ and that q_φ divides $q_1 = 4a_{11}q$. Let $b = 2lt_0^2$, $c = 2lt_0$, and let $E = (e_2, \dots, e_m)$ be a vector whose components are residue classes modulo b; let $r_\varphi(n)$ and $r_\varphi(n, E)$ denote respectively the total number of integral representations of n by the form φ, and the number of representations $n = \varphi(x_2, \dots, x_m)$ such that $x_i \equiv e_i \pmod{b}$, $i = 2, \dots, m$. Then, since lt_0^2 is representable by the form f, we may assume that $a_{11} = lt_0^2$, and on the basis of the obvious identity

$$4a_{11}f(x_1, \dots, x_m) = (2a_{11}x_1 + a_{12}x_2 + \dots + a_{1m}x_m)^2 + \varphi(x_2, \dots, x_m)$$

we obtain the following fundamental relation:

$$r_f(lt^2) = \sum_E \sum_{\substack{x=-ct \\ x\equiv EH \pmod{b}}}^{ct} r_\varphi((ct)^2 - x^2, E) = \omega_l(t) + \tilde{\omega}(t), \tag{1}$$

where the vector $H = (a_{12}, \dots, a_{1m})$;

$$\omega_l(t) = \sum_{E \neq 0} \sum_{\substack{x=-ct \\ x \equiv EH \pmod{b}}}^{ct} r_\varphi((ct)^2 - x^2, E); \qquad \tilde{\omega}(t) = \delta(t') \sum_{x=-t'}^{t'} r_\varphi((t')^2 - x^2);$$

$t' = t/t_0$; $\delta(t')$ is equal to 1 if t' is an integer, and is equal to 0 otherwise.

2°. Let $\mathfrak{M}$ be some subspace of the space of modular forms for the group $\Gamma(N)$, of dimension k and character χ, closed with respect to the Hecke operators $T(m)$, $m = 1, 2, \dots$ (see (2)); let $f_1(\tau), \dots, f_g(\tau)$ be a basis of the space $\mathfrak{M}$, and let the operator $T(m)$ be given in this basis by the matrix $\lambda(m)$. Then (2)

$$\lambda(m)\lambda(n) = \sum_{d|m,n} \lambda\left(\frac{mn}{d^2}\right) \overline{\chi}(d) d^{k-1}, \tag{2}$$

where $\overline{\chi}$ is the character conjugate to χ, and $\overline{\chi}(d) = 0$ if $(d, N) > 1$. Moreover, if

$$f_i(\tau) = \sum_{m=0}^{\infty} a_i(m) \exp(2\pi i \tau m),$$

then (2) there exist g constant matrices $B_1, \dots, B_g$, forming a basis of the commutative ring of Hecke operators on $\mathfrak{M}$, such that for $m = 1, 2, \dots$

$$\lambda(m) = \sum_{i=1}^{g} a_i(m) B_i. \tag{3}$$

Let α_{ij}^k, $i, j, k = 1, 2, \dots, g$, be the multiplication table for the matrices B_i, so that

$$B_{iB}j = \sum_{k=1}^{g} \alpha_{ij}^k B_k;$$

then from formulas (2) and (3) the following relation is obtained without difficulty:

$$a_s(mn) = \sum_{d|m,n} \sum_{i,j=1}^{g} \alpha_{ij}^s a_i\left(\frac{m}{d}\right) a_j\left(\frac{n}{d}\right) \overline{\chi}(d) d^{k-1} \mu(d), \tag{4}$$

where $m, n > 0$, and $\mu(d)$ is the Möbius function.

3°. Let $E \neq 0$. Put $\vartheta_\varphi(\tau \mid E) = \sum_{n=0}^{\infty} r_\varphi(n, E) \exp(2\pi i \tau n)$ and denote by $\mathfrak{M}(\chi)$ the space of all modular forms with respect to the group $\Gamma'(q_2)$ of character χ, of dimension $-k$, with zero constant terms of the Fourier expansions in a neighborhood of the point $i\infty$. Then from the properties of $\vartheta_\varphi(\tau \mid E)$ (see (3)) it follows that

$$\sum_{\substack{r \bmod q_2 \\ (r,q_2)=1}} \vartheta_\varphi(\tau \mid rE) \left(\frac{D_\varphi}{r}\right) \overline{\chi}(r) \in \mathfrak{M}(\chi).$$

Thus, if $f_1^\chi(\tau), \dots, f_{g_\chi}^\chi(\tau)$ is a basis of $\mathfrak{M}(\chi)$ and $\alpha_i(E,\chi)$ are the coefficients in the expansion of the last function with respect to the basis $f_i^\chi(\tau)$, then

$$\vartheta_\varphi(\tau \mid E) = \frac{1}{\varphi(q_2)} \sum_\chi \sum_{i=1}^{g_\chi} \alpha_i(E\chi)\, f_i^\chi(\tau), \tag{5}$$

where $\varphi(q_2)$ is Euler' s function. To the space $\mathfrak{M}(\chi)$, obviously, the results of the preceding item are applicable; therefore, if we denote by $\alpha_{ij}^{s\chi}$, $i, j, s = 1, 2, \dots, g_\chi$, the multiplication table for the basis of the Hecke operator ring on $\mathfrak{M}(\chi)$, then from relations (4) and (5) it follows that

$$\omega_l(t) = \frac{1}{\varphi(q_2)} \sum_\chi \sum_{i,j,s=1}^{g_\chi} \alpha_{ij}^{s\chi} \sum_{d \mid 2ct} d^{k-1} \mu(d) \left(\frac{D_\varphi}{d}\right) b_{ljs}^\chi \left(\frac{2ct}{d}\right), \tag{6}$$

where

$$b_{ijs}^\chi(2N) = \sum_{\substack{m+n=2N \\ m,n\geq 0}} a_i^\chi(m) a_j^\chi(n)\, \delta_s^\chi\left(\frac{n-m}{2}\right) \quad \text{and} \quad \delta_s^\chi(r) = \sum_{E\overline{H} \equiv r \,(\mathrm{mod}\, b)} \alpha_s(E,\chi),$$

in the formula for $\delta_s^\chi(0)$ the summation is taken over all E, $E\overline{H} \equiv 0 \pmod b$, $E \neq 0$. From relation (6) we obtain:

$$\sum_{d \mid t} d^{k-1} \left(\frac{D_\varphi}{d}\right) \omega_l\left(\frac{t}{d}\right) = \frac{1}{\varphi(q_2)} \sum_\chi \sum_{i,j,s=1}^{g_\chi} \alpha_{ij}^{s\chi} b_{ljs}^\chi(2ct). \tag{7}$$

Consider the functions

$$f_i^{\chi,r}(\tau) = \sum_{n \equiv r \,(\mathrm{mod}\, 2b)} a_i^\chi(n) \exp\left(\frac{2\pi i \tau n}{2b}\right);$$

then, since $2b$ divides q_2, it is easy to prove that for $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma'(q_2)$

$$f_i^{\chi,r}\left(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right) = \chi(\alpha) \exp\left(\frac{2\pi i r \alpha\beta}{2b}\right) (\gamma\tau+\delta)^k f_i^{\chi, r\alpha^2}(\tau). \tag{8}$$

From formula (8) and from the obvious relation

$$B_{ijs}^{\chi}(\tau) = \sum_{t=0}^{\infty} b_{ijs}^{\chi}(2ct) \exp\left(\frac{2\pi i \tau t}{t_0}\right) = \sum_{r+l\equiv 0 \,(\mathrm{mod}\, 2c)} f_i^{\chi,r}(\tau) f_j^{\chi,l}(\tau) \delta_s^{\chi}\left(\frac{l-r}{2}\right)$$

it follows that for $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma'(q_2, t_0)$

$$B_{ijs}^{\chi}\left(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right) = (\gamma\tau+\delta)^{2k} B_{ijs}^{\chi}(\tau). \tag{9}$$

4°. We now consider

$$\tilde{\omega}(t) = \delta(t') \sum_{x=-t'}^{t'} r_{\varphi}((t')^2 - x^2).$$

The series

$$\vartheta_{\varphi}(\tau) = \sum_{n=0}^{\infty} r_{\varphi}(n) \exp(2\pi i \tau n)$$

is a modular form of dimension $-k$

of the character χ_0, where $\chi_0(r) = \left(\frac{D_{\varphi}}{r}\right)$, with respect to the group $\Gamma'(q_1)$ (recall that q_{φ} divides q_1). Let $\widetilde{\mathfrak{M}}$ be the space of all modular forms with respect to $\Gamma'(q_1)$, of dimension $-k$, and of character χ_0. Let $f_1(\tau), \dots, f_g(\tau)$ be its basis, with $f_1(\tau) = \vartheta_{\varphi}(\tau)$, and let $f_2(\tau), \dots, f_g(\tau)$ have no constant terms in their Fourier expansions (in a neighborhood of the point $i\infty$). We apply to $\widetilde{\mathfrak{M}}$ the results of item 2°. Let the multiplication table of the basis of the Hecke operator ring on $\widetilde{\mathfrak{M}}$ be $\tilde{\alpha}_{ij}^{s}$, $i, j, s = 1, 2, \dots, g$; then, computing $\tilde{\omega}(t)$, we obtain:

$$\tilde{\omega}(t) = \delta(t') \left(\sum_{\substack{m+n=2t' \\ m,n>0}} r_{\varphi}(mn) + 2 \right) = \delta(t') \left(\sum_{i,j=1}^{g} \tilde{\alpha}_{ij}^{1} \sum_{d|2t'} d^{k-1} \mu(d) \left(\frac{D_{\varphi}}{d}\right) b_{ij}\left(\frac{2t'}{d}\right) \right.$$

$$\left. -2 \sum_{i=1}^{g} \tilde{\alpha}_{1i}^{1} \sum_{d|2t'} d^{k-1} \mu(d) \left(\frac{D_{\varphi}}{d}\right) a_i\left(\frac{2t'}{d}\right) + 2 \right), \tag{10}$$

where

$$b_{ij}(n) = \sum_{\substack{k+s=n \\ k,s\geq 0}} a_i(k) a_j(s).$$

From relation (10) we obtain

$$\sum_{d|t} d^{k-1} \left(\frac{D_\varphi}{d}\right) \tilde{\omega}\left(\frac{t}{d}\right) =$$

$$= \delta(t') \left(\sum_{i,j=1}^{g} \tilde{\alpha}_{ij}^{1} b_{ij}(2t') - 2 \sum_{i=1}^{g} \tilde{\alpha}_{1i}^{1} a_i(2t') + 2 \sum_{d|t} d^{k-1} \left(\frac{D_\varphi}{d}\right) \right). \tag{11}$$

We note that the series

$$B_{ij}(\tau) = \sum_{t'=0}^{\infty} b_{ij}(2t') \exp(2\pi i \tau t')$$

is a modular form of dimension $-2k$, of unit character with respect to the group $\Gamma'(q_1)$, and hence also with respect to $\Gamma'(q_2, t_0)$; as for the series

$$A_i(\tau) = \sum_{t'=0}^{\infty} a_i(2t') \exp(2\pi i \tau t')$$

and

$$G(\tau) = g_0 + \sum_{t'=1}^{\infty} \left(\sum_{d|t'} d^{k-1} \left(\frac{D_\varphi}{d}\right) \right) \exp(2\pi i \tau t'),$$

they are modular forms with respect to $\Gamma'(q_1)$ of dimension $-k$ and character $\chi_0(r) = \left(\frac{D_\varphi}{r}\right)$.

5°. Since

$$g_l(t) = \sum_{\substack{d|t \\ (d,t_0)=1}} d^{k-1} \left(\frac{2lD}{d}\right) r_f\left(\frac{lt^2}{d^2}\right) = \sum_{t|d} d^{k-1} \left(\frac{D_\varphi}{d}\right) \omega_l\left(\frac{t}{d}\right) + \sum_{d|t} d^{k-1} \left(\frac{D_\varphi}{d}\right) \tilde{\omega}\left(\frac{t}{d}\right), \tag{12}$$

then, applying to formula (12) relations (7), (9), and (11), we obtain the assertion of the theorem.

Remark. On the basis of the computation of concrete examples one may conjecture that, at least for ternary forms, the series $\widetilde{F}_l(\tau)$ entering into the formulation of the theorem is identically zero, and $G_l(\tau)$, up to a constant, is a modular form of dimension $-2k$ with respect to $\Gamma'(q_2, t_0)$.

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REFERENCES

1. A. N. Andrianov, *DAN*, **141**, No. 1, 9 (1961).
2. E. Hecke, *Math. Ann.*, a) **114** (1937); b) **114**, 316 (1937).
3. M. Eichler, *Quadratische Formen und orthogonale Gruppen*, Berlin, 1952.

Note: Figure translations are in progress. See original paper for figures.

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