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Abstract

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ON THE DYNAMIC EQUILIBRIUM OF A FOG CLOUD ABOVE THE SURFACE OF A LIQUID

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PHYSICS

The picture of the dynamic equilibrium of a cloud of supersaturated vapor above the surface of a liquid is as follows: in supersaturated vapor rising from the surface of the liquid in a convective air flow, liquid droplets are formed. Moving in the cloud, they grow all the time, i.e., absorb the excess vapor, and ultimately fall onto the surface of the liquid. Thus, in dynamic equilibrium, the entire amount of vapor coming from the surface of the liquid is absorbed by droplets present in the air.

We shall describe a fog cloud above the surface of a liquid by the distribution function $f(R, x)$ with respect to droplet sizes at a given height x and the supersaturation of the vapor $\Delta(x)$. The complete system of equations consists of the continuity equation in the space of sizes and coordinates and the substance-balance equation. The continuity equation has the form

$$\frac{\partial f}{\partial t} + \frac{\partial f \cdot v_x}{\partial x} + \frac{\partial f \cdot v_R}{\partial R} = w(R, \Delta(x)); \quad (1)$$

$w(R, \Delta(x))$ is the number of droplets of size R appearing per unit volume per unit time. It can be shown that, if surface tension and the decrease of supersaturation with height are taken into account, only droplets whose sizes exceed some critical value will be stable. We are interested in droplets considerably larger than this critical size; therefore one may assume that droplets of zero size are formed, and at the same time neglect, in the expression for the growth rate of a droplet v_R , the term associated with surface tension. Then, in the hydrodynamic approximation,

$$v_R = \frac{D}{R} \Delta; \quad D = D_0 \frac{\rho_p}{\rho_1},$$

v_R is the diffusional growth rate of a droplet, D_0 is the diffusion coefficient of the vapor; Δ is the supersaturation of the vapor; R is the droplet size; $v_x = u - \beta R^2$ is the established Stokes velocity of motion of a droplet in the cloud; u is the

velocity of the convective air flow; $\beta = \frac{2}{9} \frac{\rho_l}{\eta} g$; g is the acceleration due to gravity; η is the viscosity of air; ρ_p is the vapor density; ρ_l is the liquid density.

The substance-balance equation has the form

$$-u \frac{\partial C}{\partial x} = -u \frac{\partial \Delta}{\partial x} = 4\pi \frac{\rho_l}{\rho_p} \int_0^\infty f v_R R^2 dR + \frac{\partial \Delta}{\partial t}; \quad \Delta = C - C(T),$$

$$\Delta|_{x=0} = \Delta_0; \quad (2)$$

$C(T)$ is the equilibrium concentration of vapor.

The boundary condition $\Delta|_{x=0} = \Delta_0$ arises, for example, as a result of a temperature jump in some thin layer near the surface of the liquid; therefore the vapor, evaporating, immediately enters a region of lower temperature and becomes supersaturated: $\Delta_0 = C(T_1) - C(T_2)$; here T_1 is the temperature of the liquid, T_2 the temperature of the surrounding air.

Thus, equation (1) in the stationary case has the form

$$(u - \beta R^2) \frac{\partial f}{\partial x} + \frac{\partial}{\partial R} (f v_R) = w(R, \Delta) = n(\Delta) \delta(R). \quad (3)$$

Its solution is $f = -\frac{R}{D} \frac{n(\Delta(x_0))}{\Delta(x_0)}$; $f = 0$, $R > R_{\max}(x)$, where $x_0 = x_0(R, x)$ is an integral of the characteristic equation

$$(u - \beta R^2) dR = v_R dx. \quad (3^*)$$

The quantity $\alpha = n(\Delta)/\Delta$, generally speaking, depends strongly on the conditions in which the fog cloud is found: contamination, ionization, etc.

Let us now determine $\Delta(x)$. In order to simplify the problem, we shall take into account that at a given height the largest number is made up of droplets arriving from the upper layers of the cloud, which are sinks of vapor at this height. The trajectories of these droplets can be obtained from (3*). From this equation it is seen that a droplet, as its size increases, first rises upward while slowing down, and then descends downward. At the upper point of the trajectory its size reaches the value $R^0 = \sqrt{u/\beta}$. The distance from the place of its birth is $l_0 = u \frac{R^{02}}{4D\Delta_0}$. The path traversed by the droplet during the time of fall is of the order of the characteristic size of the cloud L . We assume that the relation $(R^0/\bar{R})^2 \ll 1$ is fulfilled, where \bar{R} is the mean droplet size. This, as we shall see, is equivalent to the condition

$$\gamma_v |\Delta_0 \rho_p| \ll 1;$$

γ_v is the amount of water in droplets per 1 cm³ of cloud. Hence it follows that the growth of the droplet during the rise may be neglected. Then the characteristic equation takes the form

$$-\frac{\beta R^2}{v_R} dR = dx, \quad \frac{\beta R^4}{4} = \int_x^{x_0} D\Delta(x) dx; \quad (4)$$

x_0 is the coordinate of the droplet' s birth.

Thus, the problem is solved to accuracy including terms of order R^0/\bar{R} .

Introduce dimensionless variables

$$\alpha(\Delta) = \frac{n(\Delta(\zeta_0))}{\Delta(\zeta_0)} \cdot \frac{\Delta_0}{n(\Delta_0)}; \quad \Delta \rightarrow \frac{\Delta}{\Delta_0};$$

$$\zeta = \frac{x}{L_0}; \quad r = \frac{R}{R^0}; \quad \varepsilon = \frac{l_0}{L_0}; \quad \frac{1}{L_0} = 4\pi \frac{\rho_l}{\rho_p} \frac{R^{03}}{u} \frac{n(\Delta_0)}{\Delta_0}.$$

In the new notation equation (2) takes the form

$$\frac{d\Delta}{d\zeta} = -\frac{\Delta}{4\varepsilon} \int_{\zeta}^{\infty} \alpha(\Delta(\zeta_0)) \Delta(\zeta_0) \frac{d\zeta_0}{\left(\frac{1}{\varepsilon} \int_{\zeta}^{\zeta_0} \Delta d\zeta\right)}; \quad \Delta|_{\zeta=0} = 1. \quad (5)$$

Before seeking a solution of this equation, it is necessary to make an assumption concerning $\alpha(\Delta)$. The simplest assumption of a linear dependence of $n(\Delta)$ above the threshold Δ_{cr} gives

$$\alpha(\Delta) = \begin{cases} 1, & \Delta > \Delta_{cr}, \\ 0, & \Delta < \Delta_{cr}. \end{cases}$$

Since $\alpha(\Delta) = 0$, $\Delta < \Delta_{cr}$, beginning from some height $\zeta = \zeta_{cr}$ there will be no stationary state, and the region where $\Delta = \Delta_{cr}$ grows monotonically with time.

For solving (5) it is convenient to introduce a new variable

$$\Phi = \frac{1}{\varepsilon} \int_{\zeta}^{\zeta_{cr}} \Delta d\zeta.$$

Then (5) takes the form

$$\frac{d\Delta}{d\Phi} = \frac{\varepsilon}{4} \int_0^{\Phi} \frac{d\Phi'}{(\Phi - \Phi')^{1/4}}; \quad \Delta|_{\Phi=0} = \Delta_{cr}, \quad (6)$$

and its solution will be

$$\Delta - \Delta_{\text{cr}} = \frac{4}{21} \varepsilon \Phi^{7/4}; \quad \xi = \frac{4}{7} \left(\frac{21}{4} \right)^{4/7} \varepsilon^{3/7} \int_{\Delta}^1 \frac{d\Delta'}{\Delta'(\Delta' - \Delta_{\text{cr}})^{3/7}}. \quad (7)$$

If the supersaturation at the base of the cloud is considerably greater than the critical one, then for the greater part of the cloud one may use the approximate expression for $\Delta(\xi)$:

$$\Delta(\xi) = \left(1 + \frac{\xi}{\frac{4}{3}(21/4)^{4/7} \varepsilon^{3/7}} \right)^{-7/3}. \quad (8)$$

The maximum droplet size at a given height is found from (4):

$$r_{\text{max}}(\xi) = \left[\frac{1}{\varepsilon} \int_{\xi}^{\xi_{\text{cr}}} \Delta d\xi \right]^{1/4} = \Phi^{1/4} = \frac{21}{4} \left(\frac{\Delta - \Delta_{\text{cr}}}{\varepsilon} \right)^{1/7}. \quad (9)$$

Our approximation is valid, as we indicated above, under the condition $(R^0/\bar{R})^2 \simeq r_{\text{max}}^{-2}(\xi) \simeq \varepsilon^{2/7} \ll 1$. Let us determine what this corresponds to. For this purpose we compute the water content at the base of the cloud (the amount of water in 1 cm³):

$$\gamma_w = \frac{4\pi}{3} \rho_l \int_0^{\infty} f R^3 dR = \frac{4}{15} \Delta_0 \rho_p r_{\text{max}}^5(0) \varepsilon \simeq \Delta_0 \rho_p \varepsilon^{2/7}.$$

Hence

$$\varepsilon^{2/7} \simeq \frac{\gamma_w}{\Delta_0 \rho_p}. \quad (10)$$

Thus, our approximation is valid when the excess water in the cloud is present chiefly in the form of vapor.

For sufficiently large supersaturation, when the behavior of the cloud over the greater part of it does not depend on the critical supersaturation, it may be taken to be zero. In this case one can find the solution of (5) also when $\alpha(\Delta) = \Delta^k$, where $k > 0$.

If $0 \leq k < 1$, it is convenient to use the variable $\Phi = \frac{1}{\varepsilon} \int_{\xi}^{\infty} \Delta d\xi$.

In the case $k \geq 1$, it is better to choose as the new variable $\Phi = \frac{1}{\varepsilon} \int_{\xi}^1 \Delta d\xi$.

In both cases the solution may be sought in the form $\Delta = A\Phi^\beta$.

As a result we obtain:

for $0 \leq k < 1$

$$\Delta(\xi) = (\varepsilon C)^{\frac{1}{1-k}} \Phi(\xi)^{\frac{7}{4} \frac{1}{1-k}}, \quad \Phi(\xi) = r_{\max}^4(\xi),$$

$$\Delta(\xi) = \left(1 + \frac{[1 - \frac{4}{7}(1-k)]C^{4/7}}{\frac{4}{7}(1-k)\varepsilon^{3/7}} \xi \right)^{-\frac{1}{1-\frac{4}{7}(1-k)}},$$

$$f(r, \xi) = \frac{R^0 n(\Delta_0)}{D\Delta_0} (\varepsilon C)^{\frac{k}{1-k}} r (r_{\max}^4(\xi) - r^4)^{\frac{7}{4} \frac{k}{1-k}},$$

$$C = \frac{1-k}{7} \frac{\Gamma\left(\frac{3/4 k+1}{1-k}\right) \Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{7}{4} \frac{1}{1-k}\right)};$$

for $k = 1$

$$\Delta(\xi) = e^{-\alpha\Phi}, \quad \alpha = \left[\frac{\varepsilon}{4} \Gamma\left(\frac{3}{4}\right) \right]^{4/7},$$

$$\Delta(\xi) = \left(1 + \frac{\alpha}{\varepsilon} \xi \right)^{-1},$$

$$f(r, \zeta) = \frac{R^0 n(\Delta_0)}{D\Delta_0} r e^{-a(r^4 + \Phi(\zeta))} = \frac{R^0 n(\Delta_0)}{D\Delta_0} \frac{r}{1 + \frac{\alpha}{\varepsilon} \zeta} e^{-\alpha r^4};$$

for $k > 1$

$$\Delta = (\varepsilon p)^{-\frac{1}{k-1}} \Phi(\zeta)^{-\frac{7}{4} \frac{1}{k-1}}, \quad p = \frac{k-1}{7} \frac{\Gamma\left(\frac{3/4+k}{k-1}\right) \Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{7}{4} \frac{k}{k-1}\right)},$$

$$\Delta(\zeta) = \left(1 + \frac{({}^4/7(k-1) + 1)p^{4/7}}{{}^4/7(k-1)\varepsilon^{3/7}} \zeta \right)^{-\frac{1}{1+{}^4/7(k-1)}}$$

$$f(r, \zeta) = \frac{R^0 n(\Delta_0)}{D\Delta_0} (\varepsilon p)^{-\frac{k}{k-1}} r (r^4 + \Phi(\zeta))^{-\frac{7}{4} \frac{k}{k-1}}.$$

The applicability of these formulas is ensured by one and the same small parameter

$$(R^0/\bar{R})^2 \cong \varepsilon^{2/7} \cong \frac{\gamma_v}{\Delta_0 \rho_p} \ll 1.$$

The formulas obtained are also applicable to light clouds, in which one may neglect the coagulation coalescence of droplets (this is possible when

$$\lambda_{\text{st}} \sim \frac{1}{n\sigma} \sim \frac{\rho_1 \bar{R}}{\gamma_v} \gg L$$

). In this case Δ_0 should be understood as the supersaturation at the base of the cloud. This supersaturation is maintained because the droplets falling through the base of the cloud enter an unsaturated atmosphere and, evaporating, return in the form of vapor by the convective air flow, where the vapor is absorbed by the growing droplets. The supersaturation itself is created by some external cause (for example, a temperature gradient), leading to a decrease in the equilibrium concentration with height.

If the transition layer is not taken into account, the supersaturation at the base of the cloud changes discontinuously. Thus, the relative supersaturation Δ/Δ_0 will be

$$\left. \frac{\Delta}{\Delta_0} \right|_{\zeta \rightarrow +0} = 1; \quad \left. \frac{\Delta}{\Delta_0} \right|_{\zeta \rightarrow -0} = 1 - \rho; \quad \rho = \frac{|C(T_1) - C(T_2)|}{\Delta_0}.$$

Usually $\rho \gg 1$. Since the velocity is proportional to the supersaturation, the ratio of the size of the transition region to the characteristic size of the cloud, l/L , will be $l/L \sim 1/\rho \ll 1$.

In conclusion, we note that we have not taken into account the temperature gradient that is created as a result of the release of heat during vapor condensation. This can be done if the temperature change ΔT in the cloud due to vapor condensation is much smaller than $\delta T = T_1 - T_2$, the temperature jump at the base of the cloud, since in this case

$$\frac{d\Delta}{dz} \gg \frac{\partial \Delta}{\partial T} \nabla T.$$

Indeed, the corresponding estimates, with the use of the equation of convective heat conduction, give

$$\frac{\Delta T}{\delta T} \cong \frac{\rho_p}{\rho_1} \frac{q_0}{4\pi c_v T} C(T) \frac{q}{kT} \ll 1,$$

where q_0 is the latent heat of vapor condensation; c_v is the heat capacity of air; $C(T) \cong e^{-q/kT}$ is the equilibrium vapor concentration.

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Note: Figure translations are in progress. See original paper for figures.

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