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Abstract

Full Text

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EXTREMAL PROPERTIES OF SOLUTIONS OF ELLIPTIC EQUATIONS

(Presented by Academician S. L. Sobolev on 13 VI 1962)

In an open domain C let us consider an equation of elliptic type

$$\mathfrak{M}u = \sum_{i,k=1}^m a_{ik} \frac{\partial^2 u}{\partial x_i \partial x_k} + \sum b_i \frac{\partial u}{\partial x_i} + cu = -f. \quad (1)$$

For $c < 0$, solutions of this equation possess the following property (Theorem 3.1 ⁽¹⁾).

If in the domain C the conditions $c < 0$, $f \geq 0$ ($f \leq 0$) are satisfied, then a regular solution of equation (1) has no points in C of negative relative minimum (positive relative maximum).

Using the results of ⁽³⁾, we shall consider analogous properties of solutions in the case where no assumptions are made concerning the relative sign-definiteness of the functions $c(x)$ and $f(x)$. In what follows we use the notation and definitions of ^(1,3).

We shall assume that the coefficients of the operator \mathfrak{M} and the right-hand side f of the equation belong to the class $C^{(0,\lambda)}$, and

$$\sum a_{ik} \alpha_i \alpha_k \geq \alpha^2 \sum \alpha_i^2, \quad \alpha \neq 0$$

in C . Let T be a closed domain belonging to the class $A^{(1,\lambda)}$, and let $T \subset C$. If the measure of the domain T is sufficiently small, then the Dirichlet problem for equation (1) and the domain T has a unique solution ^(1, Theorem 21.1.11). In the terminology of ⁽³⁾ this means that, whatever the coefficient c and the set $S \subset C$ of dimension $n < m$, there will always be found a closed domain T such that $S \subset T - \mathfrak{F}T$, and T is smaller than the first critical one: $T < T_1(c)$ ⁽³⁾.

Thus, for any coefficient c there exist subcritical domains, and the measure of the first critical domain $T_1(c)$ is different from zero.

1. A solution u may attain extremal values on certain closed manifolds S_n of n dimensions, $n = 0, 1, 2, \dots, m$. We shall call S_n a manifold of relative minimum (maximum) if $u|_{S_n} = u_0(S_n) = \text{const}$ and $u(x) > u_0$ ($u(x) < u_0$) in some m -dimensional neighborhood $O_m(S_n)$ of the manifold S_n . If

$n = m$, then we assume that $S_m < T_1(c)$. We shall also say that the function u has no minimum (maximum) in the given interval $[a, b]$, $a > b$, if $u_0(S_n) \in [a, b]$, where S_n is any manifold of relative minimum (maximum).

Set $r(d) = f + cd$, where d is some number. Under the restrictions indicated above, the following holds.

Theorem 1. If $r(d) \geq 0$ ($r(d) \leq 0$) for every $d \in [a, b]$ in the domain C , then a regular solution of equation (1) has no relative minimum (maximum) in the interval (a, b) ($[a, b]$).

We note that here no assumptions are made concerning the smallness of the coefficient c or of the domain C , and c may, in particular, be any critical one.

Proof. Suppose the contrary. Then there is a $d \in [a, b]$ such that $v = d - u > 0$ in some neighborhood $O_m(S_n)$ of the manifold of relative minimum S_n and $v|_{\mathfrak{F}O_m(S_n)} = 0$. On the other hand, choosing d sufficiently close to $u_0(S_n)$, $d > u_0(S_n)$, the neighborhood $O_m(S_n)$ can be made sufficiently small, in particular smaller than the first critical domain.

$T_1(c)$. The boundary $\mathfrak{F}O_m(S_n)$ is defined by the equation $u(x) = d$ and, by virtue of the regularity of the solution, belongs to the class $A^{(1,\lambda)}$ ((1), § 1 and (2), Theorem 111, p. 522). Under these conditions Chaplygin's theorem is applicable ((3), Theorem 2). We have

$$\mathfrak{M}v = r(d) \geq 0, \quad v|_{\mathfrak{F}O_m(S_n)} = 0$$

and $O_m(S_n)$ is less than critical. Hence by Chaplygin's theorem it follows that $v = d - u \leq 0$. The contradiction obtained proves the theorem.

If $r(d)$ changes sign in the domain C , then the analysis of the extremal properties of the solution can be carried out in separate parts of the domain C in which $r(d)$ retains a constant sign in some interval $[a, b]$.

Remark 1. The essence of the conditions imposed on the operator \mathfrak{M} is that they ensure the validity of Chaplygin's theorem for any c and, correspondingly, for a sufficiently small domain C of class $A^{(1,\lambda)}$.

2. Corollary 1. If $c \leq 0$, $f \geq 0$, then $r(d) \geq 0$ for any $d \leq 0$, and from Theorem 1 it follows that in the interval $(0, -\infty)$ there is no relative minimum, i.e. u has no points of negative relative minimum. We obtain the classical theorem cited above. Moreover, let $c \leq \beta < 0$, with f of arbitrary sign. Then $r(d) \geq 0$ for $d \leq \inf(-f/c)$ and $(a, b) = (\inf(-f/c), -\infty)$. Hence it follows, in particular, that the absolute minimum of the solution is greater than or equal to $\min[\inf(-f/c), \min u|_{\mathfrak{F}C}]$. If $f \geq \alpha > 0$, then $r(d) \geq 0$ for $d \leq -\alpha/\beta$, i.e. the relative minimum not only cannot be negative, but also does not fall below a certain positive bound equal to $-\alpha/\beta$.

Corollary 2. Let $c \geq 0$, $f \geq 0$. Then $r(d) \geq 0$ for any $d \geq 0$, i.e. the solution u has no points of positive relative minimum in C . If $c \geq \beta > 0$, with f arbitrary, then $(a, b] = (\infty, \sup(-f/c)]$. For $f \geq \alpha > 0$ the relative minimum does not exceed a certain negative bound, less than or equal to $\sup(-f/c) \leq -\alpha/\beta$.

From Corollaries 1 and 2 we see that there is a symmetry of the extremal properties of solutions corresponding to negative and positive values of the coefficient c . If $(a, b]$ is an interval without minima of the solution u_1 of equation (1) for $c_1 = c_0 \leq 0$, then $(-b, -a]$ is such an interval for u_2 , corresponding to $c_2 = -c_0$.

Corollary 3. Let $c = \lambda_k(x) \geq 0$ and let φ_k be the corresponding eigenfunction. The interval $(a, b]$ for it is equal to $(\infty, 0]$. Hence it follows, in particular, that φ_k cannot touch zero on a manifold $S_n \subset C$.

Remark 2. If, for a nonlinear elliptic equation, Chaplygin's theorem (6) holds, then the differences of its regular solutions possess the properties stated in Theorem 1. If zero is a solution of this equation, then the solutions themselves also possess such properties.

3. The extremal properties of solutions and Chaplygin's theorem for elliptic equations were considered in (4) under the condition that in the domain C there exists a function v satisfying the inequalities $v \geq 0$, $\mathfrak{M}v \leq 0$. We shall show that this condition is equivalent to the inequality $c \leq \lambda_1$.

Theorem 2. *In order that $c \leq \lambda_1$, it is necessary and sufficient that there exist a function v satisfying in the domain C the condition $v \geq 0$, $\mathfrak{M}v \leq 0$; moreover, if $\mathfrak{M}v \neq 0$, then $c < \lambda_1$.*

For $m = 1$ we obtain Theorem 2 from this.

The necessity follows from Chaplygin's theorem (3).

Sufficiency. Suppose that $c > \lambda_1$, $v \geq 0$, $\mathfrak{M}v \leq 0$. Then from Theorem Xb of A. D. Aleksandrov (4) it follows that for this c Chaplygin's theorem is valid. In particular, if

$$\mathfrak{M}u = -f \leq 0, \quad u|_{\partial C} = 0, \quad (2)$$

then $u \geq 0$. But for $c > \lambda_1$ u cannot be everywhere positive. This

is evident from the equivalence of problem (2) to the integral equation ((1), p. 84)

$$u(x) = \int_T F(x, y)\chi(y) dy + \int_T F(x, y)f(y) dy$$

with a positive kernel, and from a simple generalization of Theorem 2 of P. S. Urysohn ((9), p. 47).

Corollary 4. For $c > \lambda_1$, the solution of the problem

$$\mathfrak{M}u = -f, \quad u|_{\partial C} = \varphi \geq 0$$

cannot be positive everywhere in the domain C .

Theorem 2 and the comparison theorems (3) make it possible in practice to estimate from below the first critical value and, consequently, to verify the applicability of Chaplygin's theorem (3).

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