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Abstract

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MATHEMATICS

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ON MAPPINGS OF METRIC SPACES

(Presented by Academician P. S. Aleksandrov on 26 II 1962)

Theorem 1. In order that a T_1 -space Y have a uniform (see ⁽¹⁾) base, it is necessary and sufficient that Y be an open, continuous, and compact* image of some metric space X ; moreover, the space X can always be assumed to be zero-dimensional (in all senses) and to have the same weight as the space Y .

This theorem is adjacent to known results of V. I. Ponomarev ⁽²⁾.

Lemma 1. Under an open bicomact mapping f of a space X onto Y , an arbitrary open locally finite cover η of the space X is transformed into a point-finite cover $f\eta$ of the space Y .

Let us outline the proof of Theorem 1.

Necessity. With the aid of Lemma 1 we can, starting from some refining sequence of locally finite covers of the space X , obtain a refining sequence of point-finite covers in the space Y , and from this already follows (see ⁽¹⁾) the existence of a uniform base in the space Y .

Sufficiency. In a space Y possessing a uniform base there exists a refining sequence of point-finite covers (see ⁽¹⁾): $\varphi = \{\gamma_n\}$, $\gamma_n = \{G_\alpha \mid \alpha \in M_n\}$.

Endow the sets M_n with the discrete topology and consider the Tikhonov product

$$M = \prod_{n=1}^{\infty} M_n.$$

To a point $m_0 \in M$, $m_0 = \{\alpha_n^0\}$, we assign the point $y_0 \in Y$: $f(m_0) = y_0$, if

$$\bigcap_{n=1}^{\infty} G_{\alpha_n^0} \ni y_0,$$

and assign nothing if

$$\bigcap_{n=1}^{\infty} G_{\alpha_n} = \Lambda.$$

Denote by E the set of points of the space M for which the mapping f is defined. Then one can verify that f is a single-valued, continuous, open, compact mapping of the zero-dimensional space E onto Y .

From Theorem 1, with the aid of a theorem of P. S. Aleksandrov ⁽¹⁾, it is derived that

Theorem 2. A collectively normal space Y that is a continuous open and compact image of a metrizable space is itself metrizable.

Theorem 3. Let X be an arbitrary metric space of weight τ . Then in B_τ ** there are subspaces M_1 and M_2 such that:

- 1) $M_1 \subseteq M_2$;
- 2) there exists a mapping $f: fM_2 = X, fM_1 = X$, continuous on M_2 (and hence on M_1) and open if it is considered only on M_1 , and closed on M_2 ;
- 3) the mapping f is compact both on M_1 and on M_2 ;
- 4) if it is assumed that X is a complete metric space, then M_2 can be chosen closed in B_τ .

Finally, we note that with the aid of Lemma 1 one easily proves

Theorem 4. An open continuous bicomact image of a paracompact space is a weakly paracompact space.

Remark 1. With regard to Theorems 2 and 3 one should point to interesting results of V. Ponomarev ⁽²⁾ and E. Michael ⁽³⁾.

* A mapping $f: X \rightarrow Y$ is called compact (bicomact) if, for an arbitrary point $y \in Y$, the set $f^{-1}(y)$ is compact (bicomact).

** The space B_τ is the space of the Tikhonov product of a countable set of discrete sets of cardinality τ .

Remark 2. P. S. Aleksandrov's question: "Does there exist a nonmetrizable normal space possessing a uniform base?" can, after Theorem 1, be formulated as follows: "Will a normal space that is an open, continuous, and compact image of some metrizable space always be metrizable?"

A partial answer to this question is given by

Theorem 5. *If a regular locally finally compact space Y is the image of some metrizable space X under an open, continuous, and compact mapping, then Y itself is metrizable.*

Remark 3. In the statement of Theorem 2, compact mappings cannot be replaced by S -mappings (see ⁽²⁾). However, the following is true.

Theorem 6. *If a bicompactum Φ is an open, continuous S -image of some metrizable space X , then both Φ and X have a countable base.*

We introduce the notion of a regular mapping of a metric space. (This notion arose in connection with the notion of a weakly uniform mapping of a metric space, defined by V. Ponomarev in (2).)

Definition 1. A mapping f of a metric space X onto Y is called **regular** if for every point $y \in Y$ and its neighborhood $O_y \ni y$ there is a neighborhood $O_1y \ni y$ such that

$$\rho[f^{-1}(O_1y), X \setminus f^{-1}(O_y)] > 0.$$

Definition 2. A mapping f of a topological metrizable space (without a fixed metric) is called **regular** if there exists on X a metric with respect to which f is regular (in the sense of Definition 1).

There exist examples: 1) of a regular continuous one-to-one mapping of a metric space onto an arbitrarily bad space; 2) of a continuous two-to-one and, hence, compact mapping of a metric space X (a subset of the line) onto a metric space, which is not regular with respect to the metric prescribed on X .

Theorem 7. *A continuous mapping of a metrizable space X onto a metrizable space Y is always regular.*

In the proof of this theorem an essential role is played by

Lemma 2. *Let $\varphi = \{\gamma_n\}$ be a refining sequence of locally finite covers of a normal* space X . Then there exists on X a metric ρ such that if $\rho(x, y) < 1/n$, where n is an arbitrary natural number and x, y are arbitrary points of the space X , then there is an element $U \in \gamma_n$ for which $x \in U$ and $y \in U$. (We shall call this metric "the metric associated with the system $\varphi = \{\gamma_n\}$ ".)*

Theorem 8. *Let f be a continuous closed mapping of a metrizable space X onto a T_1 -space Y . Then, in order that the space Y be metrizable, it is necessary and sufficient that the mapping f be regular.*

Theorem 9. *Let f be a continuous open mapping of a metrizable space X onto a T_1 -space Y . Then, in order that the space Y be metrizable, it is necessary and sufficient that f be regular.*

From Theorem 8 one can derive the following

Corollary 1. Let a continuous decomposition Z of a metrizable space X into closed sets with bicompact boundary be given. Then there exists on X a metric such that for every element $\Phi \in Z$ and every closed subset F lying inside Φ , $F \subseteq \langle \Phi \rangle$, one has

$$\rho(F, X \setminus \Phi) > 0.$$

Proof of Theorem 9. Necessity. Let $\varphi = \{\gamma_n\}$ be a fundamental family of covers of the space X , defined as follows:

$$\gamma_n = \{G \mid G \text{ is open in } X, \text{ diam } G < 1/n\}.$$

* Instead of normality of X in the hypothesis of Lemma 2, one could, of course, assume only that X is regular.

We shall show that then $\psi = \{\lambda_n = f\gamma_n\}$ is a fundamental set of coverings of the space Y .

Let a point y and its neighborhood Oy be given arbitrarily. Since the mapping f is regular, there is a neighborhood O_1y of the point y for which $\rho[f^{-1}(O_1y), X \setminus f^{-1}(Oy)] > 0$. Choose N so that

$$\frac{1}{N} < \rho[f^{-1}(O_1y), X \setminus f^{-1}(Oy)].$$

Now, if $U \in \gamma_n$ and $U \cap f^{-1}(O_1y) \neq \Lambda$, then $U \subseteq f^{-1}(Oy)$. We shall show that $\lambda_N(O_1y) \subseteq Oy$. Indeed, let $V \in \lambda_n$ and $V \cap O_1y \neq \Lambda$. Then $U = f^{-1}(V) \in \gamma_n$ and, obviously,

$$U \cap f^{-1}(O_1y) = f^{-1}(V) \cap f^{-1}(O_1y) = f^{-1}(V \cap O_1y) \neq \Lambda.$$

Then, by what has been proved, $f^{-1}(V) \subseteq f^{-1}(Oy)$. Consequently, $V \subseteq Oy$, or, in other words, $\lambda_N(O_1y) \subseteq Oy$. This proves that $\{\lambda_n\}$ is a fundamental set of coverings of the space Y . By the theorem proved in ⁽⁴⁾, the space Y is metrizable. The necessity of the condition of the theorem is proved. The sufficiency is obvious.

Lemma 3. *If the spaces X and Y are metrizable, and the mapping $f : X$ onto Y is continuous and open, then there exists in X a complete sequence $\{\gamma_n\}$ of locally finite coverings of the space X , such that $\{\lambda_n\} = \{f\gamma_n\}$ is a fundamental set of coverings of the space Y .*

Let us turn to the proof of the sufficiency of the theorem. Let the conditions of the theorem be satisfied, and let the systems $\{\gamma_n\}$ and $\{\lambda_n\}$ be chosen in accordance with the lemma. Introduce in X the metric ρ associated with the complete system of locally finite coverings $\{\gamma_n\}$ (see Lemma 2). We shall show that the mapping f is regular with respect to this metric.

Indeed, let an arbitrary point $y_0 \in Y$ and its neighborhood $Oy_0 \ni y_0$ be taken. Consider the sets $O\Phi_0 = f^{-1}(Oy_0)$ and $\Phi_0 = f^{-1}(y_0)$. There is a neighborhood O_1y_0 for which, for some natural number n_0 ,

$$\lambda_{n_0}(O_1y_0) \subseteq Oy_0.$$

Then for this n_0 , for any $U \in \gamma_{n_0}$, from $U \cap f^{-1}(O_1y_0) \neq \Lambda$ it follows that $U \subseteq O\Phi_0$. Indeed, if $U \cap f^{-1}(O_1y_0) \neq \Lambda$, then $f(U) \cap O_1y_0 \neq \Lambda$. But $f(U) \in \lambda_{n_0}$, and therefore $f(U) \subseteq Oy_0$ by the choice of the neighborhood O_1y_0 . Hence $U \subseteq f^{-1}(Oy_0)$, or

$$U \cap (X \setminus f^{-1}(Oy_0)) = \Lambda.$$

We shall show that

$$\rho(X \setminus f^{-1}(Oy_0), f^{-1}(O_1y_0)) \geq \frac{1}{n_0},$$

which will complete the proof of the theorem. But this is so, for if $x_1 \in X$, $x_2 \in X$ are arbitrary points such that $x_1 \in f^{-1}(O_1 y_0)$ and $x_2 \in X \setminus f^{-1}(O y_0)$, then, by what has been proved, from $U \in \gamma_{n_0}$ it follows that either $U \cap x_1 = \Lambda$, or $U \cap x_2 = \Lambda$. Then, by the special property of the metric ρ (see Lemma 2), $\rho(x_1, x_2) \geq 1/n_0$. But in that case

$$\rho(X \setminus f^{-1}(O y_0), f^{-1}(O_1 y_0)) = \inf_{\substack{x_1 \in f^{-1}(O_1 y_0) \\ x_2 \in X \setminus f^{-1}(O y_0)}} \rho(x_1, x_2) \geq \frac{1}{n_0} > 0,$$

which means that the mapping f is regular (with respect to the metric ρ). The theorem is proved.

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Note added in proof. Recently I obtained a result which essentially generalizes Theorems 8 and 9:

Let $f : X \rightarrow Y$ be such a continuous regular mapping of a metric space X onto a T_1 -space Y that $\langle fU_\varepsilon(f^{-1}y) \rangle \ni y$, i.e., the interior of the image of every ε -neighborhood of the full preimage of an arbitrary point $y \in Y$ contains y . Then the space Y is metrizable.

Obviously, if the mapping f is open or closed, then it all the more satisfies the condition of our theorem.

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Note: Figure translations are in progress. See original paper for figures.

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