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A. D. SUKHANOV

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Abstract

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MATHEMATICAL PHYSICS

A. D. SUKHANOV

ON THE QUESTION OF THE CONNECTION BETWEEN OPERATORS IN THE HEISENBERG AND INTERACTION REPRESENTATIONS

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1. In the Hamiltonian formalism there exists the following relation (^{1,2}) between operators in the Heisenberg and interaction representations:

$$F(x) = S^{+}(\sigma, -\infty) F_{\text{in}}(x) S(\sigma, -\infty) = S(\infty, \sigma) F_{\text{out}}(x) S^{+}(\infty, \sigma), \tag{1}$$

where $F_{\text{out}}(x)$ (or $F_{\text{in}}(x)$) is an operator in the interaction representation, and $S(\infty, \sigma)$ (or $S(\sigma, -\infty)$) is the Dyson matrix (^{3}), which becomes the S-matrix as $\sigma \to -\infty$ (or $\sigma \to \infty$).

In (^{4}) it was established that the definition of the Heisenberg field $A(x)$ contains a finite arbitrariness, which may be connected both with the arbitrariness in the T-product (^{5}) for a given S-matrix in the whole momentum space, and with the arbitrariness in defining the S-matrix off the energy surface. It is natural to expect that this arbitrariness, in accordance with (1), should find some reflection in the definition of $S(\infty, \sigma)$.

Therefore it is of interest, assuming that a finite $S(\infty, \sigma)$ exists, to investigate formula (1) in the lowest orders of perturbation theory in order to clarify the nature of possible ambiguities in $S(\infty, \sigma)$. For this purpose we shall use the formal expression

$$S(\infty, \sigma) = T \exp \left\{ -i \int_{\sigma}^{\infty} H_{\text{out}}(x'; \sigma') \, dx' \right\}, \tag{2}$$

where $H_{\text{out}}(x; \sigma)$ is the effective interaction Hamiltonian (^{6}). However, the subsequent results in fact do not depend on the still unresolved question of the finiteness of expression (2), since below we use it only in first order, where this question does not arise. A preliminary study (^{6}) of it showed that, if a finite expression of the form (2) is obtained, the coefficient functions of $S(\infty, \sigma)$ will contain arbitrariness (for a given S-matrix in the whole momentum space) only beginning with the second order. One may also hope (^{6}) that for the Thirring field the arbitrariness in the coefficient functions of $S(\infty, \sigma)$ of any order is absent. At the same time (^{4}), arbitrariness in the definition of the field $A(x)$ occurs in any theory and, moreover, even in first order.

2. Let us consider the case of a neutral scalar self-interacting field. In (^{4}) an expression was obtained for the field $A(x)$ of the form

$$A(x) = T(\varphi_{\text{out}}(x)S)S^{+} =$$

$$= \varphi_{\text{out}}(x) + \int D^{\text{adv}}(x-y)J(y)\,dy + P(\Box_x)J(x), \tag{3}$$

where $P(\Box_x)$ is a polynomial in powers of $\Box_x$ with arbitrary real coefficients. If, in accordance with the generally accepted procedure, we pass in (3) from the S-matrix to $S(\infty, \sigma)$, regarding, by definition, such a passage as unambiguous, then we obtain

$$A(x) = T(S(\infty,\sigma)\varphi_{\text{out}}(x)S(\sigma,-\infty))S^{+} = S(\infty,\sigma)\varphi_{\text{out}}(x)S^{+}(\infty,\sigma). \tag{4}$$

Expanding formally the right-hand side of (4), being guided here by the known rules for expanding expressions of the form $S\varphi_{\text{out}}(x)S^{+}$, we obtain, to first order,

$$A(x) = \varphi_{\text{out}}(x) + e\int D^{\text{adv}}(x-y)j(y)\,dy. \tag{5}$$

Thus, although the transition from (3) to (4) leads to a formula of the form (1), the formal handling of $A(x)$ of the form (4) leads to the loss of the product in the expression for the field (see (5)).

The simplest way to eliminate the contradiction between (3) and (5) is to include the product in the coefficient functions $S(\infty, \sigma)$ of first order by replacing $ej(x)$ by $e[j(x) - P(\Box_x)K_x j(x)]$, where $K_x \equiv \Box_x - m^2$. Then, instead of (5), from (4) we obtain

$$A(x) = \varphi_{\text{out}}(x) + e\int D^{\text{adv}}(x-y)j(y)\,dy + eP(\Box_x)j(x), \tag{6}$$

which is in agreement with the first order of (3). It is not hard to see, however, that the presence in $S(\infty, \sigma)$ of such a product corresponds to different definitions of the S-matrix off the energy surface, i.e., to only one possible interpretation$^{(4)}$ of the product in the definition of the field $A(x)$. Meanwhile the question of greatest interest is what product is possible in $S(\infty, \sigma)$ for a fixed definition of the S-matrix in the entire momentum space, which would correspond to the possibility of different definitions of the T-product in (3)$^{(4)}$. Such a product cannot be included in the coefficient functions $S(\infty, \sigma)$ of first order, so that it is necessary to clarify the meaning of the formal transition from (4) to (5).

The point is that the S-matrix is defined in the entire momentum space if its coefficient functions are integrable in some class of smooth functions. But when the results of multiplying the S-matrix by other quantities, for example by $\varphi_{\text{out}}(x)$, are expanded, products of generalized functions may arise which are not always integrable in the same class of functions. In particular, it is known that products of the form $S\varphi_{\text{out}}(x)$ are defined, and for the S-matrix a product arises only when expanding products of the form $T(S\varphi_{\text{out}}(x))^{(4)}$.

However, in the case of the Dyson matrix it is necessary to take into account that its coefficient functions are integrable in a narrower class of functions. Therefore, already when expanding the product $S(\infty, \sigma)\varphi_{\text{out}}(x)$ a product arises, since here we encounter the multiplication of $D^{-}(y-x)$ by $\theta(y^0-x^0)$ (for simplicity we take the surface σ to be plane). In other words, if the definition of $\varphi_{\text{out}}(y)\varphi_{\text{out}}(x)$ is fixed, then for the indeterminate expression appearing in the expansion of (4) we may introduce the definition

$$\theta(y^0 - x^0)\varphi_{\text{out}}(y)\varphi_{\text{out}}(x) = \frac{1}{i}\theta(y^0 - x^0)D^{-}(y - x) + \frac{1}{2i}P(\Box_x)\delta(x - y). \quad (7)$$

If it is used in expanding the right-hand side of (4), then without difficulty we obtain expression (6), which agrees with (3).

Thus, the product noted in $^{(4)}$ in the definition of the field (for fixed S-matrix) in first order of perturbation theory and for all theories can be connected only with a product of the form (7) in the definition of the ordinary contraction between $S(\infty, \sigma)$ and $\varphi_{\text{out}}(x)$.

3. Let us now turn to consideration of the relation of the form (1) between the interaction Hamiltonians in two representations. According to $^{(5)}$, the Hamiltonian in the Heisenberg representation has the form

$$H(x;1) = -T\left(S\int \left.\frac{\delta\mathcal{L}(y;g)}{\delta g(x)}\right|_{g=1} dy\right)S^{+} \equiv T\left(SH_{\text{out}}(x;1)\right)S^{+}, \quad (8)$$

where $\mathcal{L}(y;g)$ is the effective interaction Lagrangian $^{(5)}$.

Since H$(x;1)$, in contrast to the field $A(x)$, is a quantity connected with experiment, it makes sense to consider (8) only when the T-product is fixed. Here, however, another interesting problem arises. It is easy to see that $\mathrm{H}_{\text{out}}(x;1)$ does not coincide with the effective Hamiltonian obtained in (^{6}) in the interaction representation,

$$\mathrm{H}_{\text{out}}(x;\sigma) = -\lim_{g\to\theta_\sigma}\int \frac{\delta\mathcal{L}(y;g)}{\delta g(x)}\, g'(x)\, dx^0 dy,$$

on which $S(\infty,\sigma)$ in (2) depends. Thus it turns out that several Hamiltonians in the interaction representation may correspond to one $\mathrm{H}(x;1)$, which would be strange if one did not allow the existence of a certain operation connected with multiplication by $S(\infty,\sigma)$.

It should be noted in this connection that the expressions for $\mathrm{H}_{\text{out}}(x;1)$ and $\mathrm{H}_{\text{out}}(x;\sigma)$ are not completely equivalent. For example, $\mathrm{H}_{\text{out}}(x;1)$ contains no terms depending quadratically on the normals, and cannot stand in the Tomonaga–Schwinger equation (^{1}) for theories with derivative couplings (^{7}). Moreover, if one admits the existence of an expression for $S(\infty,\sigma)$ of the form (2), then using in it $\mathrm{H}_{\text{out}}(x;1)$ instead of $\mathrm{H}_{\text{out}}(x;\sigma)$ would lead to a violation of the internal closure of Bogoliubov' s method (^{5}), for then, even to obtain a finite S-matrix, one would have to use a definition of the T-product different from that already fixed in (8).

On the other hand, transforming (8) by analogy with the transformation of expression (3), we arrive (in accordance with (1)) at the expression

$$\mathrm{H}(x;1) = S(\infty,\sigma)\,\mathrm{H}_{\text{out}}(x;1)\,S^{+}(\infty,\sigma), \tag{9}$$

from which the impression is created that the "true" Hamiltonian in the interaction representation is $\mathrm{H}_{\text{out}}(x;1)$. However, from the preceding it is clear that expression (9) is undefined until the rules for multiplying $S(\infty,\sigma)$ by $\mathrm{H}_{\text{out}}(x;1)$ have been fixed in it.

In this connection let us consider (9) to accuracy through second order, for which it is sufficient to use $S(\infty,\sigma)$ to first-order accuracy. Then it is not difficult to show that one and the same expression for $\mathrm{H}(x;1)$ can be obtained by consistently choosing the Hamiltonian in the interaction representation and the rules for multiplying $S(\infty,\sigma)$ by it.

It should only be emphasized that if above, in the case of the field $A(x)$, it was sufficient to define the multiplication of $\underbrace{\varphi_{\text{out}}(y)\varphi_{\text{out}}(x)}$ by the θ-function, for example in the form (7), then in the present case, even if we somehow define $\theta(y^0-x^0)\,\underbrace{\varphi_{\text{out}}(y)\varphi_{\text{out}}(x)}$, products of a larger number of ordinary contractions by a θ-function will still remain undefined.

As an example let us consider a theory with $\mathcal{L}(x) = e : \varphi^4(x) :$. Then from (8) we have

$$\mathrm{H}(x;1) = \mathrm{H}_{\text{out}}(x;1) + i\int \left[\mathcal{L}(x)\mathcal{L}(y) - T(\mathcal{L}(x)\mathcal{L}(y))\right] dy. \tag{10}$$

In order to obtain an analogous expression from (9), it is necessary to introduce definitions of the form

$$\theta(y^0-x^0)\,\underbrace{\varphi_{\text{out}}(y)\varphi_{\text{out}}(x)} = \frac{1}{i}\theta(y^0-x^0)D^-(y-x) \tag{11}$$

and so on. However, even for formula (9) with $\mathrm{H}_{\text{out}}(x;1)$ as the Hamiltonian in the interaction representation, such simple definitions cannot always be used. In particular, in scalar electrodynamics $\mathrm{H}(x;1)$ contains the term $2e^2 : \varphi^*(x)\varphi(x)[n_\alpha A_\alpha(x)]^2 :$ by virtue of the corresponding definition of the T-product (^{7}). In this case, in order to obtain $\mathrm{H}(x;1)$ from (9), it is necessary to use the definition

$$\theta(y^0-x^0)\,\underbrace{\frac{\partial\varphi}{\partial y^\alpha}\frac{\partial\varphi^*}{\partial x^\beta}} = \frac{1}{i}\theta(y^0-x^0)\frac{\partial^2 D^-(y-x)}{\partial y^\alpha\partial x^\beta} - \frac{1}{2i}n_\alpha n_\beta\delta(x-y). \tag{12}$$

An even greater arbitrariness must be introduced into the definitions of products of contractions in the case where, in a formula of the form (9), $H_{\text{out}}(x;1)$ is replaced by $H_{\text{out}}(x;\sigma)$. Only formula (11) remains valid. To define the product of two contractions on the θ-function, however, it is necessary to use the formula

$$\theta(y^0-x^0)\,\underbrace{\varphi_{\text{out}}(y)\varphi_{\text{out}}(x)}\,\underbrace{\varphi_{\text{out}}(y)\varphi_{\text{out}}(x)} = -\frac{1}{i^2}\theta(y^0-x^0)D^-(y-x)D^-(y-x)+\frac{1}{4i}B:\varphi^2(x)\varphi^2(y):\delta(x-y), \tag{13}$$

where B is a logarithmically divergent coefficient $^{(5)}$. Analogous formulas also hold for products of three and four contractions of fields on the θ-function. Let us only note that, when defining expressions of the form $\theta(y^0-x^0)\varphi_{\text{out}}(x)\varphi_{\text{out}}(y)$, the additional terms must be introduced with the opposite sign, which also applies to formula (7).

Thus, both $H_{\text{out}}(x;1)$ and $H_{\text{out}}(x;\sigma)$ may be used in a formula of the form (9). Consequently, at least in the second order, the concern disappears that the presence in $H_{\text{out}}(x;\sigma)$ of counterterms $^{(6)}$ might lead to divergences in observable quantities. Such quantities include $H(x;1)$, and not $H_{\text{out}}(x;\sigma)$, and although the coefficient functions $S(\infty,\sigma)$ are assumed finite, while $H_{\text{out}}(x;\sigma)$ contains $^{(6)}$ divergent coefficients, we can always define the rules for multiplying $S(\infty,\sigma)$ by $H_{\text{out}}(x;\sigma)$ in such a way that $H(x;1)$ of the form (9) is finite.

4. Thus, it has been established that if the S-matrix in the entire momentum space is specified uniquely, then $S(\infty,\sigma)$, irrespective of the possible arbitrariness in the coefficient functions of higher orders $^{(6)}$, contains a significant arbitrariness associated with the possibility of fixing in different ways the rules for multiplying $S(\infty,\sigma)$ by operators in the interaction representation.

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Mathematical Institute named after V. A. Steklov
Academy of Sciences of the USSR

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