MATHEMATICS

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1962

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Abstract

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MATHEMATICS

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STURM' S THEOREM AND THE OSCILLATION OF SOLUTIONS OF STRONGLY ELLIPTIC SYSTEMS

(Presented by Academician I. G. Petrovsky, 26 VII 1961)

A classical area of the qualitative investigation of an ordinary differential equation of the second order is the field of questions concerning the existence of oscillatory and nonoscillatory solutions, Sturm' s theorems. At the present time analogous questions have also been studied for equations of higher orders and systems of ordinary differential equations.

For elliptic equations of the second order, Sturm' s theorems were obtained by Picone ([1]), and then repeated and supplemented in works ([2–6]). In the present paper Sturm' s theorem is proved and the question of the oscillation of solutions of strongly elliptic systems of the second order is investigated.

1. Let us consider strongly elliptic systems of the second order:

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(A_{ij}(x) \frac{\partial u}{\partial x_j} \right) + C(x) u = 0; \tag{1}$$

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(G_{ij}(x) \frac{\partial v}{\partial x_j} \right) + H(x) v = 0, \tag{2}$$

where $A_{ij}(x) = A_{ji}(x)$, $G_{ij}(x) = G_{ji}(x)$ are twice continuously differentiable, and $C(x), H(x)$ are continuous real functional symmetric square matrices of order N, defined in a domain D. It is assumed that for every nonzero column ξ of height nN and every point $x \in D$ the conditions

$$\xi^* \mathfrak{A}(x) \xi > 0, \qquad \xi^* \mathfrak{G}(x) \xi > 0,$$

hold, where

$$\mathfrak{A}(x) = \| A_{ij}(x) \|_{i,j=1}^{n}, \qquad \mathfrak{G}(x) = \| G_{ij}(x) \|_{i,j=1}^{n}.$$

Comparison theorem. *If in the domain D the conditions*

$$\xi^*(\mathfrak{A}(x) - \mathfrak{G}(x))\xi \geq 0, \tag{3}$$

$$\eta^*(H(x) - C(x))\eta \geq 0, \tag{4}$$

are satisfied, where η *is a column of height* N*, and there exists a vector solution* $u(x)$*, not identically zero, of system (1), taking zero values on the boundary* S*, then the determinant of any square matrix* $V(x)$ *of order* N*, for which*

$$V^*(x)\left[\sum_{j=1}^{n} G_{ij}(x)\frac{\partial V(x)}{\partial x_j}\right] = \left[\sum_{j=1}^{n} \frac{\partial V^*(x)}{\partial x_j} G_{ij}(x)\right] V(x) \quad (i = 1, \dots, n) \tag{5}$$

and the form

$$V^*(x)\left[\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(G_{ij}(x)\frac{\partial V(x)}{\partial x_j}\right) + H(x)V(x)\right] \tag{6}$$

is negative semidefinite, must vanish at least at one point in the domain D*.*

Proof. For the vector solution $u(x)$ of system (1) considered in the theorem, the formula

$$\int_D^{(n)} \left\{\sum_{i,j=1}^{n} \frac{\partial u^*(x)}{\partial x_i} A_{ij}(x)\frac{\partial u(x)}{\partial x_j} - u^*(x)C(x)u(x)\right\} dx = 0. \tag{7}$$

holds.

We prove the theorem by contradiction. Suppose that the determinant $V(x)$ does not vanish in D, i.e., $V^{-1}(x)$ exists. Then, for the vector $h(x) = V^{-1}(x)u(x)$, we have

$$0 \geq h^*(x)\left\{V^*(x)\left[\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(G_{ij}(x)\frac{\partial V(x)}{\partial x_j}\right) + H(x)V(x)\right]\right\} h(x)$$

$$= u^*(x)\left[\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(G_{ij}(x)\frac{\partial V(x)}{\partial x_j}\right)\right] V^{-1}(x)u(x) + u^*(x)H(x)u(x). \tag{8}$$

Integrate (8) over the domain D and add it to (7). Then, after transformations using condition (5), we obtain

$$\int_D^{(n)} \left\{ \sum_{i,j=1}^{n} \frac{\partial u^*(x)}{\partial x_i} (A_{ij}(x) - G_{ij}(x)) \frac{\partial u(x)}{\partial x_j} + u^*(x)(H(x) - C(x))u(x) + \right. \quad (9)$$

$$\left. + \sum_{i,j=1}^{n} \left(\frac{\partial u^*(x)}{\partial x_i} - u^*(x) V^{-1*}(x) \frac{\partial V^*(x)}{\partial x_i} \right) G_{ij}(x) \left(\frac{\partial u(x)}{\partial x_j} - \frac{\partial V(x)}{\partial x_j} V^{-1}(x) u(x) \right) \right\} dx \leq 0.$$

The contradiction obtained with conditions (3), (4) proves the theorem.

Two vector solutions $v(x)$ and $w(x)$ of system (2) will be called conjugate if everywhere in D

$$v^*(x) \sum_{j=1}^{n} G_{ij}(x) \frac{\partial w(x)}{\partial x_j} - w^*(x) \sum_{j=1}^{n} G_{ij}(x) \frac{\partial v(x)}{\partial x_j} \equiv 0 \quad (i = 1, \dots, n). \qquad (10)$$

Definition 1. A system of n vector solutions of system (2) that are pairwise conjugate and linearly independent will be called a **conjugate system of solutions**.

Now conditions (5) and (6) may be replaced by the requirement that $V(x)$ be a conjugate system of solutions.

The definition introduced by us is a direct generalization of the corresponding notion from the theory of systems of ordinary differential equations. This theorem can be applied to obtain various criteria for unique solvability of the Dirichlet problem (cf. (6)); in doing so it is natural to choose systems (2) of a simpler type, for example of the type indicated below.

The question of the existence of a conjugate system of solutions in the general case remains open and requires further investigation. However, for systems (2) in which the matrices $H(x) = H(x_k)$, $G_{kk}(x) = G_{kk}(x_k)$, $G_{ik}(x) = 0$ $(i \neq k)$ for at least one index k, a conjugate system of solutions may be sought in the form of a matrix $V(x) = V(x_k)$. Then $V(x_k)$ must be a conjugate system of solutions of the system of ordinary differential equations

$$\frac{d}{dx_k} \left(G_{kk}(x_k) \frac{dV}{dx_k} \right) + H(x_k) V = 0,$$

existence of which is known (7). In particular, for strongly elliptic systems with constant coefficients without mixed derivatives, adjoint systems of solutions always exist.

2. Definition 2. We shall call system (1) **nonoscillatory** in the domain D if, for every subdomain D, the first boundary-value problem is uniquely solvable, i.e., if the system has no nontrivial solutions that vanish on the boundary of an n-dimensional domain belonging to D.

Theorem 2. If in the domain D there exist symmetric functional square matrices $\Phi_1, \dots, \Phi_n$ of order N, continuous in all arguments and continuously differentiable respectively with respect to the arguments $x_1, \dots, x_n$, such that the matrix

$$-C(x) + \sum_{j=1}^{n} \frac{\partial \Phi_j(x)}{\partial x_j} - \Phi^*(x)\mathfrak{A}^{-1}(x)\Phi(x), \tag{11}$$

where $\Phi^*(x) = (\Phi_1(x), \dots, \Phi_n(x))$, and $\mathfrak{A}^{-1}(x)$ is the inverse matrix to $\mathfrak{A}(x)$, is positive definite, then system (1) will be nonoscillatory in the domain D.

In the case of ordinary differential equations and systems of the second order, Theorem 2 coincides with the necessary and sufficient conditions for nonoscillation of M. I. El' shin (8) and Sternberg (9), and in the case of a single elliptic equation—with the condition of V. Ya. Skorobogat'ko (3). We shall show further that, for a single equation of elliptic type, the theorem gives not only sufficient but also necessary conditions for nonoscillation. We have not succeeded in proving the necessity of this condition in the case of systems, but it is probably valid.

3. Let us consider in more detail the case of a single elliptic equation of second order

$$L(u) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + C(x)u = 0, \qquad a_{ij}(x) = a_{ji}(x). \tag{12}$$

Theorem 3. For nonoscillation of the solutions of equation (12) in the domain D, it is necessary and sufficient that there exist in D a positive twice continuously differentiable function $v(x)$ such that $L(v) \leqslant 0$.

The proof, analogous to that given in (10), uses the fact that the solutions of equation (12) have no isolated nodal points (5). The substitution

$$\varphi_i(x) = -\frac{1}{v}\sum a_{ij}(x)\frac{\partial v}{\partial x_j}$$

proves the necessity of the condition of Theorem 2.

For concrete applications it is convenient to replace the necessary and sufficient conditions of Theorems 2 and 3 by sufficient but more easily verifiable ones.

Corollary 1. If the solutions of the equations

$$\frac{d^2y}{dx_i^2} + p_i(x_i)y = 0$$

are nonoscillatory (oscillatory) in the domains D_i, then the solutions of the equation

$$\Delta u + c(x)u = 0,$$

where Δ is the n-dimensional Laplace operator,

$$c(x) \leqslant \sum_{i=1}^{n} p_i(x_i) \quad \left(c(x) \geqslant \sum_{i=1}^{n} p_i(x_i) \right),$$

will be nonoscillatory (oscillatory) in the domain D, which is the direct product of the domains D_i.

This corollary makes it possible to generalize to elliptic equations the known criteria of nonoscillation due to Kneser, Lyapunov, Bellman, Hill, and others. For lack of space they are not given here.

Corollary 2. If the solutions of the equations

$$\Delta u + c_j(x)u = 0 \quad (j = 1, \dots, m)$$

are nonoscillatory in the domain D, then the solutions of the equation

$$\Delta u + \sum_{j=1}^{m} a_j c_j(x)u = 0$$

are also nonoscillatory in the domain D, for $a_j = \text{const} > 0$, $\sum_{j=1}^{m} a_j \leqslant 1$.

This corollary shows that the functions $c(x)$ for which the equation $\Delta u + c(x)u = 0$ has nonoscillatory solutions in the domain D form a convex set in the space of continuous functions.

Received
22 VII 1961

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Note: Figure translations are in progress. See original paper for figures.