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Abstract

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ON RIESZ BASES IN $\mathcal{L}_2(0,1)$

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A sequence $\varphi_1(x), \dots, \varphi_n(x), \dots,\ x \in (0,1)$, from $\mathcal{L}_2(0,1)$ is called a **basis** in $\mathcal{L}_2(0,1)$ if for every element $f(x) \in \mathcal{L}_2(0,1)$ there exists a unique expansion

$$f = c_1\varphi_1 + \dots + c_n\varphi_n + \cdots, \tag{1}$$

converging to f in the mean.

Following N. K. Bari (^{1}), we shall call the basis $\varphi_1, \dots, \varphi_n, \dots$ a **Riesz basis** if there exists a constant $\gamma \geq 1$ such that, for every $f(x) \in \mathcal{L}_2(0,1)$, the inequalities

$$\gamma^{-1} \sum |c_k|^2 \leq \|f\|^2 \leq \gamma \sum |c_k|^2 \tag{2}$$

hold.

Consider the ordinary linear differential operator of order n

$$l(y) = y^{(n)} + p_1(x)y^{(n-1)} + \dots + p_n(x)y, \tag{3}$$

defined on the interval $[0,1]$. We shall assume that the coefficients $p_s(x)$ are summable on $(0,1)$ together with their derivatives up to order $(n{-}s)$, $s = 1, \dots, n$. We are interested in the problem of eigenfunctions of the operator $l(y)$ under the boundary conditions

$$U_\nu(y) \equiv U_{\nu 0}(y) + U_{\nu 1}(y) = 0, \qquad \nu = 1, \dots, n, \tag{4}$$

$$U_{\nu 0}(y) = \alpha_\nu y_0^{(k_\nu)} + \alpha_{\nu,k_\nu-1} y_0^{(k_\nu-1)} + \cdots; \quad U_{\nu 1}(y) = \beta_\nu y_1^{(k_\nu)} + \beta_{\nu,k_\nu-1} y_1^{(k_\nu-1)} + \cdots;$$

$$|\alpha_\nu| + |\beta_\nu| \neq 0; \qquad n{-}1 \geq k_1 \geq k_2 \geq \dots \geq k_n; \qquad k_{\nu+2} < k_\nu; \qquad y_0^{(s)} = \left.\frac{d^s y(x)}{dx^s}\right|_{x=0}, \quad y_1^{(s)} = \left.\frac{d^s y(x)}{dx^s}\right|_{x=1}.$$

The system adjoint to $\{\varphi_k\}$ is the system of eigenfunctions $\{\varphi_k^*\}$ of the operator $l^*(y)$ (adjoint to $l(y)$) under the boundary conditions (4^*), adjoint to the conditions (4). (We note that φ_k^* corresponds to the eigenvalue $\lambda_k^* = \overline{\lambda}_k$, where λ_k is the eigenvalue corresponding to φ_k.) In what follows, the functions φ_k and φ_k^* will be assumed normalized so that $(\varphi_k, \varphi_j^*) = \delta_{kj}$, $k, j = 1, 2, \ldots$

The conditions (4) are called **regular** ($^{2-4}$) if certain determinants θ_0 and θ_1 for odd n, and θ_{-1}, θ_1 for even n, are nonzero $((^4), p.51)$. We note that $\theta_0, \theta_1, \theta_{-1}$, and hence also the regularity of the conditions (4), depend only on the coefficients α_ν and β_ν, $\nu = 1, \ldots, n$, at the highest derivatives in the conditions (4).

We shall say that the conditions (4) are **strongly regular** if they are regular and, in addition, for even n, $\theta_0^2 \neq 4\theta_1\theta_{-1}$.

In (^{3}) it was in fact established that the root functions $\{\varphi_k\}$ of the operator $l(y)$ under strongly regular boundary conditions (4) form a basis in $\mathcal{L}_2(0,1)$.

The purpose of the present note is to prove the following assertion:

Theorem. The totality of root functions of the operator (3) under stren—

regular boundary conditions (4), forms a Riesz basis in the space $\mathcal{L}_2(0,1)$.

Since, under the hypotheses of the theorem, the eigenvalues (except for a finite number of them) are simple, it is clearly sufficient to carry out the proof of the theorem for the case when all eigenvalues are simple and, thus, the root functions coincide with the eigenfunctions.

In (^{5,6,10}) more general problems were considered for certain operators. As applied to our problem, the results of these papers, as well as of (^{11}), are contained in the theorem formulated above as special cases.

Let us note that the theorem remains valid also in the case when the coefficients $l(y)$ in (3) are, as in Birkhoff (^{2}), analytic functions of λ, and when, as in Tamarkin (^{3}), certain integral operators are added to the boundary conditions (4).

The proof of the theorem is based on several lemmas.

Lemma 1. *If the conditions (4) for the operator $l(y)$ are regular (strongly regular), then the adjoint conditions (4^*) for the adjoint operator $l^*(y)$ are also regular (strongly regular).*

By a direct verification we see that the principal coefficients in the conditions (4^*) are determined only by the principal coefficients of (4). Therefore the proof of Lemma 1 can be carried out by comparing the asymptotic formulas for the eigenvalues λ_k and λ_k^* as $|k| \to \infty$ (^{4}).

Consider, for $f(\xi) \in \mathcal{L}_2(0,\infty)$, the Hilbert transform (^{7})

$$F(x) = \int_0^\infty \frac{f(\xi)\, d\xi}{\xi - x}, \tag{5}$$

which is an analytic function of x for $\operatorname{Im} x > 0$ and for $\operatorname{Im} x < 0$. Denote by $\|F\|^2_{\mathcal{L}_2(0,\infty e^{i\psi})}$ the integral

$$\int_0^\infty |F(\rho e^{i\psi})|^2 \, d\rho.$$

Lemma 2. *For every ψ, $F(x) \in \mathcal{L}_2(0, \infty e^{i\psi})$, and for $\psi \equiv 0 \pmod{\pi}$*

$$\|F\|_{\mathcal{L}_2(0,\infty e^{i\psi})} = \|f\|_{\mathcal{L}_2(0,\infty)},$$

while for $\psi \not\equiv 0 \pmod{\pi}$

$$\|F\|_{\mathcal{L}_2(0,\infty e^{i\psi})} \le \sqrt{\frac{\pi}{|\sin\psi|}} \, \|f\|_{\mathcal{L}_2(0,\infty)}.$$

If $\psi \equiv 0 \pmod{\pi}$, then Lemma 2 is known (^{1}). For what follows it is convenient to introduce into consideration the kernel

$$K_N(x,\xi) = \sum_{k=0}^{N} e^{-(\alpha+i\beta)xk-(\gamma+i\delta)\xi k}, \tag{6}$$

where N is some natural number; $\alpha, \beta, \gamma, \delta$ are real numbers, $\alpha \ge 0$, $\gamma \ge 0$, and β and δ are such that, when $\alpha = 0$, $\beta = 2\pi$ or $\beta = -2\pi$, and when $\gamma = 0$, $\delta = 2\pi$ or $\delta = -2\pi$, $x \in [0,1]$, $\xi \in [0,1]$.

Let $f(x) \in \mathcal{L}_2(0,1)$. Then

$$F_N(x) = \int_0^1 K_N(x,\xi) f(\xi) \, d\xi$$

is an analytic function of x for every N.

Lemma 3. *There exists a constant C_0, independent of N and of $f(\xi)$, such that*

$$\|F_N\| \le C_0 \|f\|. \tag{7}$$

If $\alpha = \gamma = 0$, and $\beta = \pm 2\pi$, $\delta = \pm 2\pi$, then

$$F_N(x) = c_0 + c_1 e^{2\pi i x} + \cdots + c_{Ne}^{2\pi i N x},$$

where $c_k = (f, e^{\mp 2\pi i k \xi})$. Inequality (7) in this case is⋯

is a simple consequence of Bessel' s inequality for trigonometric series. Let $\alpha > 0,\ \gamma > 0$. Represent $F_N(x)$ in the form

$$F_N(x) = \int_0^1 \frac{f(\xi)\,d\xi}{1 - e^{-\theta(x,\xi)}} - \int_0^1 \frac{f(\xi)e^{-(N+1)\theta(x,\xi)}}{1 - e^{-\theta(x,\xi)}}\,d\xi = F^{(1)}(x) - F_N^{(2)}(x); \quad (8)$$

$$F^{(1)}(x) = \int_0^\varepsilon \frac{f(\xi)\,d\xi}{1 - e^{-\theta(x,\xi)}} + \int_\varepsilon^1 \frac{f(\xi)\,d\xi}{1 - e^{-\theta(x,\xi)}} = \Phi_{1\varepsilon}(x) + \Phi_{2\varepsilon}(x), \quad (9)$$

where

$$\theta(x,\xi) = -(\alpha + i\beta)x - (\gamma + i\delta)\xi = -(\gamma + i\delta)(\xi - Axe^{i\psi}), \quad A = \left|\frac{\alpha + i\beta}{\gamma + i\delta}\right|,$$

$\psi = \arg(\alpha + i\beta) - \arg(\gamma + i\delta) + \pi$, and ε is for the time being an arbitrary number from $(0,1)$. We note that $\theta(x,\xi) = 0$ only when $x = \xi = 0$ (since $\alpha > 0,\ \gamma > 0$). It is verified directly that

$$\|\Phi_{2\varepsilon}\| \le \|f\|/\sqrt{1 - e^{-\varepsilon\gamma}}, \qquad \|\Phi_{1\varepsilon}\|_{\mathcal{L}_2(\varepsilon,1)} \le \|f\|/\sqrt{1 - e^{-\varepsilon\alpha}}. \quad (10)$$

Now take $\varepsilon > 0$ so small that for $x \le \varepsilon,\ \xi \le \varepsilon$ the inequality

$$|1 - e^{-\theta(x,\xi)} - \theta(x,\xi)| \le |\theta^2(x,\xi)|$$

holds. Then, according to (10) and Lemma 2,

$$\|\Phi_{1\varepsilon}\|^2 = \|\Phi_{1\varepsilon}\|^2_{L_2(0,\varepsilon)} + \|\Phi_{1\varepsilon}\|^2_{L_2(\varepsilon,1)} \le \left(1 + \frac{1}{1 - e^{-\varepsilon\alpha}} + \frac{\pi}{(\gamma^2 + \delta^2)\sin\psi}\right)\|f\|^2 \quad (11)$$

(when applying Lemma 2 we extended the function $f(\xi)$ from the interval $(0,\varepsilon)$ to the interval $(0,\infty)$ by zero). Taking (10) and (11) into account, we obtain from (9) estimate (7) for $F^{(1)}(x)$. The estimate for $F_N^{(2)}(x)$, uniform in N, is obtained analogously.

Let now $\alpha > 0,\ \gamma = 0$ (similarly, $\alpha = 0,\ \gamma > 0$); then $\theta(x,\xi)$ in (9) vanishes at two points of the square $0 \le x \le 1,\ 0 \le \xi \le 1$: at the point $x = \xi = 0$ and at the point $x = 0,\ \xi = 1$ (similarly at the points $x = \xi = 0,\ x = 1,\ \xi = 0$). To obtain estimate (7) in this case, we first divide the interval $0 \le \xi \le 1$ into two parts $(0,1/2)$ and $(1/2,1)$. In each of these parts the required estimate (7) is obtained by the method described above for the case $\alpha > 0,\ \gamma > 0$.

Let

$$\widetilde{K}_N(x,\xi) = \sum_{k=1}^{N} \frac{1}{k} e^{-(\alpha+i\beta)xk-(\gamma+i\delta)\xi k}, \tag{12}$$

where $\alpha, \beta, \gamma, \delta, N$ are the same constants as in the kernel (6).

Lemma 4. Let $f(\xi) \in \mathcal{L}_2(0,1)$. Then

$$\widetilde{F}_N(x) = \int_0^1 \widetilde{K}_N(x,\xi) f(\xi)\, d\xi$$

belongs to $\mathcal{L}_2(0,1)$ for every N, and the estimate

$$\|\widetilde{F}_N\| \le C_1 \|f\| \tag{13}$$

holds, where the constant C_1 does not depend on N and $f(x)$.

Consider now the kernel

$$R_N(x,\xi) = \sum_{|k|\le N} \varphi_k^*(x)\varphi_k^*(\xi), \tag{14}$$

consisting of eigenfunctions of the operator $l^*(y)$ under the conditions (4^*).

Lemma 5. Let $f(x) \in \mathcal{L}_2(0,1)$. Then

$$g_N(x) = \int_0^1 R_N(x,\xi) f(\xi)\, d\xi$$

for every N belongs to $\mathcal{L}_2(0,1)$, and the estimate

$$\|g_N\| \le C_2 \|f\|$$

holds, where the constant C_2 does not depend on N and $f(\xi)$.

Assume, for definiteness, that the order of the operator (3) is odd, $n = 2\mu - 1$. For the normalized eigenfunctions $\varphi_k^*(x)$ of the adjoint problem, by virtue of Lemma 1 one can write, uniformly with respect to $x \in [0,1]$, the asympto-

asymptotic in k as $|k| \to \infty$ expressions

$$\varphi_k^*(x) = \sum_{|s|<\nu-1} \left(A_s^{\pm} + \frac{B_s^{\pm}}{k}\right) e^{\omega_s(2k+\sigma)x} \sum_{\substack{|s|\ge\nu \\ s\ne\nu}} \left(A_s^{\pm} + \frac{B_s^{\pm}}{k}\right) e^{\omega_s(2\pi k+\sigma)(1-x)} +$$

$$+\left(A_\nu^\pm + \frac{B_\nu^\pm}{k}\right) e^{(2k\pi i+\sigma)x} + O\left(\frac{1}{k^2}\right), \tag{15}$$

where

$$\omega_s = \exp\left[i\pi\left(1+\frac{1}{2n}-\frac{2s}{n}\right)\right], \qquad s = 0, \pm 1, \dots;$$

σ is a complex number; A_s^+ and A_s^- (B_s^+ and B_s^-) are certain constant numbers which should be substituted in (15) in place of $A_s^\pm$ ($B_s^\pm$) for $k > 0$ and $k < 0$, respectively; $B_\nu^\pm \neq 0$. By direct verification we see that the kernel $R_N(x,\xi)$ is represented in the form

$$R_N(x,\xi) = \sum_{s=1}^{M} [D_{sK} N^{(s)}(x,\xi) + E_{sK} N^{(s)}(1-x,\xi) + G_{sK} N^{(s)}(x,1-\xi) +$$

$$+ H_{sK} N^{(s)}(1-x,1-\xi) + \widetilde{D}_s \widetilde{K}_N^{(s)}(x,\xi) + \widetilde{E}_s \widetilde{K}_N^{(s)}(1-x,\xi) +$$

$$+ \widetilde{G}_s \widetilde{K}_N^{(s)}(x,1-\xi) + \widetilde{H}_s \widetilde{K}_N^{(s)}(1-x,1-\xi)] + \widetilde{\widetilde{K}}_N(x,\xi), \tag{16}$$

where M is some natural number; $D_s, \dots, \widetilde{H}_s$ are certain complex numbers, while the functions $K_N^{(s)}(x,\xi)$ and $\widetilde{K}_N^{(s)}(x,\xi)$ are kernels of the type (6) and (12), respectively; the kernel $\widetilde{\widetilde{K}}_N(x,\xi)$ is uniformly bounded with respect to N and uniformly continuous in x, ξ.

Lemma 5 now follows immediately from Lemmas 3 and 4 with the aid of representation (16).

To prove the theorem it remains for us to use a theorem of N. K. Bari (^{1}).

Bari' s theorem. *In order that the basis $\{\varphi_k\}$ be a Riesz basis, it is necessary and sufficient that there exist a bounded, invertible, Hermitian, positive operator A taking the system $\{\varphi_k\}$ into its conjugate system $\{\varphi_k^*\}$.*

Thus, suppose that the operator A is given on the basis by: $A\varphi_k = \varphi_k^*$, $k = 1, 2, \dots$. By this it is defined for any

$$f_N = c_{-N}\varphi_{-N} + \dots + c_N\varphi_N, \qquad N = 0, 1, \dots,$$

$$c_k = (f, \varphi_k^*), \qquad Af_N = \sum_{k=-N}^{+N} c_k \varphi_k^* = \int_0^1 R_N(x,\xi) f(\xi)\, d\xi,$$

where $R_N(x, \xi)$ is the kernel (14). By virtue of Lemma 5, the operator A extends to all of $\mathcal{L}_2(0,1)$, and $\|A\| \leq C_2$. Let f and $g \in \mathcal{L}_2(0,1)$,

$$f = c_0\varphi_0 + \cdots + c_N\varphi_N + \cdots, \qquad g = b_0\varphi_0 + \cdots + b_N\varphi_N + \cdots,$$

then

$$(Af, g) = c_0\bar{b}_0 + \cdots + c_N\bar{b}_N + \cdots = (f, Ag),$$

i.e. A is a Hermitian operator.

Moreover,

$$(Af, f) = |c_0|^2 + \cdots + |c_N|^2 + \cdots > 0,$$

and $(Af, f) = 0$ only under the condition $f = 0$. Thus the operator A satisfies all the conditions of Bari' s theorem. Hence the basis $\{\psi_k\}$ is a Riesz basis.

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Note: Figure translations are in progress. See original paper for figures.

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